Algebraic Geometry Fall 2018 Homework 1

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Due Wednesday Aug 29 (start of class)

Throughout A will denote a commutative ring with identity.

Problem 1. Describe $\operatorname{spec}(A)$ as a set for each of the following rings. Try to both explictly describe the members and draw an intuitive picture.

(a) $A = \mathbb{R}[x]$

(b) $A = \mathbb{F}_2[x]$

(c) $A = \mathbb{Z}/60\mathbb{Z}$ (Hint: Chinese remainder theorem)

Problem 2. Recall that for every subset $S \subseteq A$, the vanishing set of S is defined to be

$$V(S) = \{\mathfrak{p} \in \operatorname{spec}(A) : S \subseteq \mathfrak{p}\}$$

and that every Zariski open subset is the complement of a set of this form. Prove that the collection of Zariski open sets $\tau = \{V(S)' : S \subseteq A\}$ defines a topology on the set $\operatorname{spec}(A)$. Specifically show

(a) $\varnothing$ and $\operatorname{spec}(A)$ are elements of τ

(b) Any union of elements of τ belongs to τ

(c) Any finite intersection of elements of τ belongs to τ

(Hint: use the fact that $V(S) = V(I)$ for I the ideal of A generated by S)

Problem 3. Give (with proof) an example of a ring A with $\operatorname{spec}(A)$ connected but reducible.

Problem 4. Consider the prime ideal $\mathfrak{p} = (y - x^2)$ of $A = \mathbb{C}[x, y]$.

1. Explain why the closure $\overline{\{\mathfrak{p}\}}$ of $\{\mathfrak{p}\}$ in $\operatorname{spec}(A)$ is $V(\mathfrak{p})$ (equivalently, $\mathfrak{p}$ is a generic point of $V(\mathfrak{p})$).
2. Prove that $V(\mathfrak{p}) = \{(x - a, y - a^2) : a \in \mathbb{C}\} \cup \{\mathfrak{p}\}$.