Algebraic Geometry Fall 2018 Homework 2

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Due Wednesday Sept 12 (start of class)

Problem 1. Consider the set of 2×3 circulant, complex-valued matrices of rank at most 1:

$$V = \left\{ A = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_1 & a_2 \end{pmatrix} \in M_{3,2}(\mathbb{C}) : \mathrm{rk}(A) \leq 1 \right\}.$$

We can identify V with a subset of $\mathbb{A}^4_{\mathbb{C}}$ by sending

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_1 & a_2 \end{pmatrix} \mapsto (a_1, a_2, a_3, a_4).$$

(a) Show that V is an algebraic set by showing that it is the zero set of a collection of three homogeneous polynomials of degree 2 in $R = \mathbb{C}[x_1, x_2, x_3, x_4]$.

(b) Show that $I(V)$ is the ideal of R generated by the polynomials from (a).

(c) Show that the affine coordinate ring $R/I(V)$ of V is an integral domain and therefore that V is itself an affine variety.

(d) Show that the dimension of V is 2. This is interesting, since we showed V to be the intersection of three hypersurfaces in $\mathbb{A}^4$. Are there two hypersurfaces in $\mathbb{A}^4$ whose intersection is V? In other words, can V be expressed as the zero set of *two* polynomials in R?

Problem 2. Let X and Y be topological spaces, and for each open subset U of X let

$$h_Y(U) = \{\text{continuous functions } f : U \to Y\}.$$

and for $V \subseteq U \subseteq X$ let

$$\mathrm{res}_{U,V} : h_Y(U) \to h_Y(V), \quad f \mapsto f|_V.$$

Show that h_Y is a sheaf.

Problem 3. Let X and Y be topological spaces, $f : X \to Y$ a continuous function, and $\mathcal{F}$ is a presheaf on X. Show that

$$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$$

with the obvious restriction map is a presheaf on Y. Show that if $\mathcal{F}$ is a sheaf, so is $f_*\mathcal{F}$. This (pre)sheaf is called the push-forward of $\mathcal{F}$.

Problem 4. Let X be a topological space, that $\mathcal{F}$ is a presheaf on X and that $\mathcal{G}$ is a sheaf on X. Show that

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) := \{\text{sheaf morphisms } \mathcal{F}|_U \to \mathcal{G}|_U\}$$

with the obvious restriction map is a sheaf. Note that here $\mathcal{F}|_U$ denotes $\mathcal{F}$ restricted to the topological space U, and similarly for $\mathcal{G}$. The sheaf $\mathcal{H}om$ is called "sheaf Hom".