Algebraic Geometry Fall 2018 Homework 2

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Due Wednesday Sept 12 (start of class)

Problem 1. Consider the set of 2×3 circulant, complex-valued matrices of rank at most 1:

$$V = \left\{ A = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_1 & a_2 \end{pmatrix} \in M_{3,2}(\mathbb{C}) : \text{rk}(A) \leq 1 \right\}.$$

We can identify V with a subset of $\mathbb{A}^4_{\mathbb{C}}$ by sending

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_1 & a_2 \end{pmatrix} \mapsto (a_1, a_2, a_3, a_4).$$

(a) Show that V is an algebraic set by showing that it is the zero set of a collection of three homogeneous polynomials of degree 2 in $R = \mathbb{C}[x_1, x_2, x_3, x_4]$.

(b) Show that $I(V)$ is the ideal of R generated by the polynomials from (a).

(c) Show that the affine coordinate ring $R/I(V)$ of V is an integral domain and therefore that V is itself an affine variety.

(d) Show that the dimension of V is 2. This is interesting, since we showed V to be the intersection of three hypersurfaces in $\mathbb{A}^4$. Are there two hypersurfaces in $\mathbb{A}^4$ whose intersection is V? In other words, can V be expressed as the zero set of *two* polynomials in R?

Solution 1.

(a) The rank of the 2×3 matrix

$$A = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_1 & x_2 \end{pmatrix} \in M_{3,2}(\mathbb{C})$$

is less than 2 if and only if *all* of the 2×2 minors of A vanish. Thus V is the subset of $\mathbb{A}^4_{\mathbb{C}}$ determined by

$$V = Z(x_1^2 - x_2x_4, x_2^2 - x_1x_3, x_1x_2 - x_3x_4).$$

(b) Let J be the ideal generated by the three homogeneous polynomials from (a). SInce $Z(J) = V$, we know that $J \subseteq I(V)$ and so it suffices to prove $I(V) \subseteq J$. One may readily check that any monomial in $\mathbb{C}[x_1, x_2, x_3, x_4]$

is equivalent modulo J to a monomial of the form $x_3^m x_4^n$, $x_1 x_3^m x_4^n$, or $x_2 x_3^m x_4^n$. Thus any f is of the form

$$f = \sum_{m,n} (a_{mn} x_3^m x_4^n + b_{mn} x_1 x_3^m x_4^n + c_{mn} x_2 x_3^m x_4^n) \mod J. \tag{1}$$

If $f \in I(V)$, then for any $s, t \in \mathbb{C}$ the point $(ts^2, t^2 s, t^3, s^3) \in V$ and therefore

$$0 = f(ts^2, t^2 s, t^3, s^3) = \sum_{m,n} \left(a_{mn} t^{3m} s^{3n} + b_{mn} t^{3m+1} s^{3n+2} + c_{mn} t^{3m+2} s^{3n+1} \right)$$

The expression on the right hand side is a polynomial in s and t whose zero set is every point of $\mathbb{A}^2_{\mathbb{C}}$, so by Weak Nullstellensatz it is the zero polynomial. It follows that the coefficients a_{mn}, b_{mn}, c_{mn} are all identically zero. Hence $f(x_1, x_2, x_3, x_4) = 0 \mod J$ ie. $f(x_1, x_2, x_3, x_4) \in J$ and this proves $I(V) \subseteq J$.

(c) Taking our cue from (b), we consider the ring homomorphism

$$\varphi : \mathbb{C}[x_1, x_2, x_3, x_4] \to \mathbb{C}[y_1, y_2], \; f(x_1, x_2, x_3, x_4) \mapsto f(y_1 y_2^2, y_1^2 y_2, y_1^3, y_2^3).$$

If $f \in J$, then since $(ts^2, t^2 s, t^3, s^3) \in V$ for all $s, t \in \mathbb{C}$, we know that the zero set of $f(y_1 y_2^2, y_1^2 y_2, y_1^3, y_2^3)$ is $\mathbb{A}^2_{\mathbb{C}}$ and therefore $f(y_1 y_2^2, y_1^2 y_2, y_1^3, y_2^3)$ is identically zero by weak Nullstellensatz. Conversely, if $f(y_1 y_2^2, y_1^2 y_2, y_1^3, y_2^3) = 0$, then by the same argument made in (b) we know $f \in J$. Hence $\ker(\varphi) = J$ and thus φ defines an injection of the ring of regular functions $A(V) = \mathbb{C}[x_1, x_2, x_3, x_4]/I(V)$ into $\mathbb{C}[y_1, y_2]$. In particular since $\mathbb{C}[y_1, y_2]$ is an integral domain, so too is $A(V)$ and consequently $I(V)$ is a prime ideal and V is a variety.

(d) The image B of $A(V)$ in $\mathbb{C}[y_1, y_2]$ is generated by $y_1^3, y_1^2 y_2, y_1 y_2^2, y_2^3$ and therefore the fraction field $K(B)$ of B is $\mathbb{C}(y_2/y_1, y_1^3)$. This is an extension of $\mathbb{C}$ of transcendence degree 2, so the Krull dimension of B (and therefore $A(V)$) is 2. Hence the dimension of Y is 2.

The variety V can be generated by two polynomials. To see this, consider the following 2×2 and 2×3 matrices A and B:

$$A = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_1 & x_2 \end{pmatrix}, \quad B = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_1 & x_2 \\ x_2 & x_3 & 0 \end{pmatrix}.$$

If $x_1^2 = x_2x_4$ then B is nonsingular if and only if A has rank 2. Consequently V is the zero set of $\det(B)$ and $x_1^2 - x_2x_4$, ie

$$V = Z(x_1^2 - x_2x_4, (x_2^2 - x_3x_1)x_2 - (x_1x_2 - x_3x_4)x_3).$$

This shows that V is the intersection of two hypersurfaces in $\mathbb{A}^4_{\mathbb{C}}$ but, not ideal-wise (or scheme-wise for that matter).

Problem 2. Let X and Y be topological spaces, and for each open subset U of X let

$$h_Y(U) = \{\text{continuous functions } f : U \to Y\}.$$

and for $V \subseteq U \subseteq X$ let

$$\text{res}_{U,V} : h_Y(U) \to h_Y(V), \quad f \mapsto f|_V.$$

Show that h_Y is a sheaf.

Solution 2. Note that $h_Y(\varnothing) = \{\varnothing\}$, and that for any $f : U \to Y$ we have $\text{res}_{U,U}(f) = f|_U = f$ and $\text{res}_{V,W}(\text{res}_{U,V}(f)) = \text{res}_{V,W}(f|_V) = (f|_V)|_W = f_W = \text{res}_{U,W}(f)$ so that h_Y is a presheaf.

If $U = \bigcup_i V_i$ and $f, g : U \to Y$ satisfy $f|_{V_i} = g|_{V_i}$ for all i, then $f = g$, so we have uniqueness. Furthermore, if $f_i : V_i \to Y$ is continuous for all i and $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i \neq j$ then the function $f : U \to Y$ defined by $f(x) = f_i(x)$ if $x \in U_i$ is well-defined. Furthermore, it's continuous since for any $x \in U$ there is a neighborhood U_i of x where f agrees with the continuous function f_i. Thus we have gluing and this shows $\mathcal{F}$ is a sheaf.

Problem 3. Let X and Y be topological spaces, $f : X \to Y$ a continuous function, and $\mathcal{F}$ is a presheaf on X. Show that

$$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$$

with the obvious restriction map is a presheaf on Y. Show that if $\mathcal{F}$ is a sheaf, so is $f_*\mathcal{F}$. This (pre)sheaf is called the push-forward of $\mathcal{F}$.

Solution 3. The map $U \mapsto f^{-1}(U)$ defines a covariant functor $G : \text{Open}(Y) \to \text{Open}(X)$ and $f_*\mathcal{F}$ is precisely the composition of functors $\mathcal{F} : \text{Open}(X) \to \text{Sets}$, ie. $f_*\mathcal{F} = \mathcal{F} \circ G$. In particular $f_*\mathcal{F}$ is a contravariant functor $\text{Open}(X) \to \text{Sets}$, ie. a presheaf.

Now assume $\mathcal{F}$ is a sheaf. Suppose that $V = \bigcup_i V_i$ for some open sets V, V_i of Y and let $U_i = f^{-1}(V_i)$ and $U = f^{-1}(V)$. If $s, t \in f_*\mathcal{F}(V) = \mathcal{F}(U)$ with $\text{res}_{VV_i}(s) = \text{res}_{VV_i}(t)$ for all i then since $\text{res}_{VV_i} = \text{res}_{UU_i}$ we have $\text{res}_{UU_i}(s) = \text{res}_{UU_i}(t)$ for all i. Since $\mathcal{F}$ is a sheaf, it follows that $s = t$. Similarly, given $s_i \in f_*\mathcal{F}_i(V_i)$ with $\text{res}_{V_i,V_i \cap V_j}(s_i) = \text{res}_{V_j,V_i \cap V_j}(s_j)$ for all $i \neq j$ then we have $\text{res}_{U_i,U_i \cap U_j}(s_i) = \text{res}_{U_i,U_i \cap U_j}(s_j)$ for all $i \neq j$. Since $\mathcal{F}$ is a sheaf, it follows that there exists $s \in \mathcal{F}(U)$ such that $\text{res}_{U,U_i}(s) = s_i$ for all i. Hence $\text{res}_{V,V_i}(s) = s_i$ for all i. This proves that we have uniqueness and gluing for $f_*\mathcal{F}$ and so it is a sheaf.

Problem 4. Let X be a topological space, that $\mathcal{F}$ is a presheaf on X and that $\mathcal{G}$ is a sheaf on X. Show that

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) := \{\text{sheaf morphisms } \mathcal{F}|_U \to \mathcal{G}|_U\}$$

with the obvious restriction map is a sheaf. Note that here $\mathcal{F}|_U$ denotes $\mathcal{F}$ restricted to the topological space U, and similarly for $\mathcal{G}$. The sheaf $\mathcal{H}om$ is called "sheaf Hom".

Solution 4. Just do the usual thing.