Algebraic Geometry Fall 2018 Homework 3

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Due Wednesday Sept 19 (start of class)

Problem 1. Let (X, τ) be a topological space and let $\beta \subseteq \tau$ be a basis for τ. A sheaf on the basis β is a function

$$\mathcal{F} : \beta \to \text{Sets}$$

along with a collection of functions $\text{res}_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ for all $U, V \in \beta$ with $V \subseteq U$ satisfying the following five axioms.

(0) $\mathcal{F}(\varnothing) = \text{singleton}$ if $\varnothing \in \beta$

(1) $\text{res}_{U,U} = \text{identity}$ for all $U \in \beta$

(2) $\text{res}_{U,W} = \text{res}_{V,W} \circ \text{res}_{U,V}$ for all $U, V, W \in \beta$

(3) if $U \in \beta$ and $\{U_i\}$ is an open covering of U by elements of β and if $f, g \in \mathcal{F}(U)$ with $\text{res}_{U,U_i}(f) = \text{res}_{U,U_i}(g)$ for all i, then $f = g$

(4) if $U \in \beta$ and $\{U_i\}$ is an open covering of U by elements of β and if $f_i \in \mathcal{F}(U_i)$ with $\text{res}_{U_i,W}(f_i) = \text{res}_{U_j,W}(f_j)$ for all i, j and $W \in \beta$ with $W \subseteq U_i \cap U_j$, then there exists $f \in \mathcal{F}(U)$ with $\text{res}_{U,U_i}(f) = f_i$ for all i

Prove that any sheaf on the basis β extends uniquely to a sheaf on X.

Problem 2. Let R be a ring, let M be an R-module, and let $X = \text{spec}(R)$ with the Zariski topology. For any $r \in R$ define

$$\widetilde{M}(D(r)) = \{s^{-1}m : \ m \in M, s \in R, \ \text{with } Z(\{s\}) \subseteq Z(\{r\})\}$$

and for $r, s \in R$ with $D(s) \subseteq D(r)$ let $\text{res}_{D(r),D(s)}$ be the natural map.

(a) Prove that $\widetilde{M}(D(r)) \cong M_r$ for all $r \in R$

(b) Prove that $\widetilde{M}$ defines a sheaf on the basis $\beta = \{D(r) : r \in R\}$ of the Zariski topology, and therefore extends uniquely to a sheaf $\widetilde{M}$ on X

(c) Prove that the stalk of $\widetilde{M}$ at a point $\mathfrak{p} \in \text{spec}(R)$ is $M_{\mathfrak{p}}$

Note that in the case $M = R$, the construction $\widetilde{R}$ is the structure sheaf $\mathcal{O}_X$.

Problem 3. Let $X = \text{spec}(R)$ and $Y = \text{spec}(S)$ be affine schemes. Prove that there is a bijective correspondence

$$\{\text{morphisms of schemes } X \to Y\} \longleftrightarrow \{\text{homomorphisms of rings } S \to R\}.$$

Show that this correspondence sends isomorphisms to isomorphisms.

Problem 4. Let X be a scheme and for all $p \in X$ let $\mathfrak{m}_p$ denote the maximal ideal of the local ring $(\mathcal{O}_X)_p$ (the stalk of the structure sheaf at p). For any $f \in \mathcal{O}_X(X)$ we let f_p denote the image of f under the natural map $\mathcal{O}_X(X) \to (\mathcal{O}_X)_p$ and define

$$X_f = \{p \in X : \ f_p \notin \mathfrak{m}_p\}.$$

(a) Show that X_f is an open subscheme of X

(b) If $X = \text{spec}(R)$ and $r \in \mathcal{O}_X(X) = R$, show that $X_r = D(r)$ and is affine (in fact it is isomorphic to $\text{spec}(A_r)$ as a scheme)

(c) Show that an open subset of an affine scheme is not necessarily affine (eg. $\text{spec}(\mathbb{C}[x,y]) \backslash \{(x,y)\}$)

(d) Show that if Y is an affine scheme, then there exist global sections $s_1, \ldots, s_r \in \mathcal{O}_Y(Y)$ with Y_{s_i} affine for all i and $s_1, \ldots, s_r$ generates the unit ideal on $\mathcal{O}_Y(Y)$.

BONUS: Prove that (d) is actually an if and only if condition.