Algebraic Geometry Fall 2018 Homework 3

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Due Wednesday Sept 19 (start of class)

Problem 1. Let (X, τ) be a topological space and let $\beta \subseteq \tau$ be a basis for τ. A sheaf on the basis β is a function

$$\mathcal{F} : \beta \to \text{Sets}$$

along with a collection of functions $\text{res}_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ for all $U, V \in \beta$ with $V \subseteq U$ satisfying the following five axioms.

(0) $\mathcal{F}(\varnothing) = \text{singleton}$ if $\varnothing \in \beta$

(1) $\text{res}_{U,U} = \text{identity}$ for all $U \in \beta$

(2) $\text{res}_{U,W} = \text{res}_{V,W} \circ \text{res}_{U,V}$ for all $U, V, W \in \beta$

(3) if $U \in \beta$ and $\{U_i\}$ is an open covering of U by elements of β and if $f, g \in \mathcal{F}(U)$ with $\text{res}_{U,U_i}(f) = \text{res}_{U,U_i}(g)$ for all i, then $f = g$

(4) if $U \in \beta$ and $\{U_i\}$ is an open covering of U by elements of β and if $f_i \in \mathcal{F}(U_i)$ with $\text{res}_{U_i,W}(f_i) = \text{res}_{U_j,W}(f_j)$ for all i, j and $W \in \beta$ with $W \subseteq U_i \cap U_j$, then there exists $f \in \mathcal{F}(U)$ with $\text{res}_{U,U_i}(f) = f_i$ for all i

Prove that any sheaf on the basis β extends uniquely to a sheaf on X.

Solution 1. Without loss of generality, we take $\varnothing \in \beta$. The extension of $\mathcal{F}$ may be defined in terms of limits. For each $U \in \tau \backslash \beta$, define

$$\mathcal{F}(U) = \lim_{V \in \beta, V \subseteq U} \mathcal{F}(V).$$

By this we mean that $\mathcal{F}(U)$ is a set with a collection of maps $\text{res}_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ satisfying the condition that if $W, V \in \beta$ with $W \subseteq V$ then $\text{res}_{U,W} = \text{res}_{V,W} \circ \text{res}_{U,V}$. Moreover, $\mathcal{F}(U)$ is universal in the sense that if S is another set with a collection of maps $f_V : S \to \mathcal{F}(V)$ for all $V \in \beta$ satisfying $f_W = \text{res}_{V,W} \circ f_V$ for all $V, W \in \beta$ with $W \subseteq V$, then there exists a unique map $g_U : S \to \mathcal{G}(U)$ satisfying $f_V = \text{res}_{U,V} \circ g_U$ for all $V \in \beta$ with $V \subseteq U$. Note that the choice of set $\mathcal{F}(U)$ and maps $\text{res}_{U,V}$ is not unique, but is unique up to unique isomorphism by the universal property. Therefore in our extension of $\mathcal{F} : \beta \to \text{Sets}$ to $\mathcal{F} : \tau \to \text{Sets}$ we are at this point choosing a fixed limit for each $U \in \tau \backslash \beta$. If $U, V \in \tau \backslash \beta$ we also define $\text{res}_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ to be the unique map obtained from the universal property above.

Note that if we want to construct $\mathcal{F}(U)$ very concretely, we can take an open cover $\{V_i\} \subseteq \beta$ of U and define

$$\mathcal{F}(U) = \{(s_i) \in \prod_i \mathcal{F}(V_i) : \forall W \in \beta,\ W \subseteq V_i \cap V_j \text{ we have } \mathrm{res}_{V_i,W}(s_i) = \mathrm{res}_{V_j,W}(s_j)\}.$$

We can explicitly construct the restriction map accordingly.

Obviously $\mathcal{F}(\varnothing) =$ singleton by (0). Moreover, by the uniqueness part of the universal property or (1), $\mathrm{res}_{U,U} = \mathrm{id}_U$ for all $U \in \tau$. Furthermore, by the universal property or (2), $\mathrm{res}_{U,W} = \mathrm{res}_{V,W} \circ \mathrm{res}_{U,V}$ for all $U, V, W \in \tau$ with $W \subseteq V \subseteq U$. Thus $\mathcal{F}$ is a presheaf.

To prove that $\mathcal{F}$ is a sheaf, we require two observations which follow immediately from the universal property of a limit. First is that if $s, t \in \mathcal{F}(U)$ with $\mathrm{res}_{U,V}(s) = \mathrm{res}_{U,V}(t)$ for all $V \in \beta$ with $V \subseteq U$ then $s = t$. The second is that if for all $V \in \beta$ with $V \subseteq U$ we have an $s_V \in \mathcal{F}(V)$ and $\mathrm{res}_{V,W}(s_V) = s_W$ for all $W, V \in \beta$ with $W \subseteq V \subseteq U$, then there exists $s \in \mathcal{F}(U)$ such that $\mathrm{res}_{U,V}(s) = s_V$ for all $V \in \beta$.

Suppose $s, t \in \mathcal{F}(U)$ and there exist $\{U_i\} \subseteq \tau$ covering U such that $\mathrm{res}_{U,U_i}(s) = \mathrm{res}_{U,U_i}(t)$ for all i. Then for all $V \subseteq U$, we can cover each $U_i \cap V$ with a collection of $\{V_{ij}\} \subseteq \beta$ and then $\mathrm{res}_{V,V_{ij}}(\mathrm{res}_{U,V}(s)) = \mathrm{res}_{V,V_{ij}}(\mathrm{res}_{U,V}(t))$ for all i, j so that $\mathrm{res}_{U,V}(s) = \mathrm{res}_{U,V}(t)$ for all $V \subseteq U$ by property (3). It follows that $s = t$.

Next, suppose that $U \in \tau$ and $\{U_i\} \subseteq \tau$ covers U and that there exists $s_i \in \mathcal{F}(U_i)$ such that $\mathrm{res}_{U_i,U_i \cap U_j}(s_i) = \mathrm{res}_{U_j,U_i \cap U_j}(s_j)$ for all $i \neq j$. Then for any $V \subseteq U$ we can cover $U_i \cap V$ with $\{V_{ij}\} \subseteq \beta$. Set $s_{ij} = \mathrm{res}_{U_i,V_{ij}}(s_i)$ and note that for all $W \subseteq V_{ij} \cap V_{k\ell}$

$$\begin{aligned}\mathrm{res}_{V_{ij},W}(s_{ij}) &= \mathrm{res}_{U_i,W}(s_i) = \mathrm{res}_{U_i \cap U_k,W}(\mathrm{res}_{U_i,U_i \cap U_k}(s_i)) \\ &= \mathrm{res}_{U_i \cap U_k,W}(\mathrm{res}_{U_k,U_i \cap U_k}(s_k)) = \mathrm{res}_{U_k,W}(s_k) = \mathrm{res}_{V_{k\ell},W}(s_{k\ell}).\end{aligned}$$

Therefore since the V_{ij} cover V, (4) tells us there exists $s_V \in V$ satisfying $\mathrm{res}_{V,V_{ij}}(s_V) = s_{ij}$ for all i, j. If $V, W \in \beta$ with $W \subseteq V \subseteq U$, then $s_W = \mathrm{res}_{V,W}(s_V)$ and therefore there exists $s \in U$ such that $\mathrm{res}_{U,V}(s) = s_V$ for all $V \in \beta$ with $V \subseteq U$. The uniqueness result of the previous paragraph implies that $\mathrm{res}_{U,U_i}(s) = s_i$ for all i. This proves $\mathcal{F}$ is a sheaf.

Now if $\mathcal{G}$ is any other sheaf on X satisfying $\mathcal{G}(V) = \mathcal{F}(V)$ for all $V \in \beta$ then the sheaf properties of $\mathcal{G}$ imply that $\mathcal{G}(U)$ will be a limit of $\mathcal{F}(V)$ for $V \in \beta$ with $V \subseteq U$. Therefore for all U there will exist a unique bijection $\mathcal{F}(U) \to \mathcal{G}(U)$ which is compatible with restriction. This is obviously a sheaf isomorphism, so the extension $\mathcal{F}$ is unique up to unique isomorphism.

Problem 2. Let R be a ring, let M be an R-module, and let $X = \operatorname{spec}(R)$ with the Zariski topology. For any $r \in R$ define

$$\widetilde{M}(D(r)) = \{s^{-1}m : \; m \in M, s \in R, \text{ with } Z(\{s\}) \subseteq Z(\{r\})\}$$

and for $r, s \in R$ with $D(s) \subseteq D(r)$ let $\operatorname{res}_{D(r),D(s)}$ be the natural map.

(a) Prove that $\widetilde{M}(D(r)) \cong M_r$ for all $r \in R$

(b) Prove that $\widetilde{M}$ defines a sheaf on the basis $\beta = \{D(r) : r \in R\}$ of the Zariski topology, and therefore extends uniquely to a sheaf $\widetilde{M}$ on X

(c) Prove that the stalk of $\widetilde{M}$ at a point $\mathfrak{p} \in \operatorname{spec}(R)$ is $M_{\mathfrak{p}}$

Note that in the case $M = R$, the construction $\widetilde{R}$ is the structure sheaf $\mathcal{O}_X$.

Solution 2.

(a) The module $\widetilde{M}(D(r))$ is M localized at the multiplicative set $S = \{s : Z(\{s\}) \subseteq Z(\{r\})\}$. There is a natural R-module homomorphism $M_r \to \widetilde{M}(D(r))$, induced by the inclusion $\{r^n : n > 0\} \subseteq S$.

Now suppose that m/r^n maps to zero in $\widetilde{M}(D(r))$. Then $sm = 0$ for some $s \in S$ and it follows that for some $j > 0$ and $a \in R$ that $r^j m = asm = 0$ and therefore $m/r^n = 0$ in M_r. Thus the map $M_r \to \widetilde{M}(D(r))$ is injective. Next, suppose $m/s \in \widetilde{M}(D(r))$. Again we may write $r^j = as$ for some s, j and it follows that am/r^j maps to m/s in $\widetilde{M}(D(r))$. Thus the map is an isomorphism.

(b) Properties (0-2) are obvious from the definitions. Property (3-4) may be reduced to the case that $U = \operatorname{spec}(R)$ and $\{U_i\} = \{D(f_i)\}_{i=1}^r$ with $(f_1, \ldots, f_r) = R$. They then follow from the exactness of the sequence of R-modules

$$0 \to M \to \prod_i M_{f_i} \xrightarrow{(m_i) \mapsto (m_i - m_j)} \prod_{i \neq j} M_{f_i f_j}.$$

(c) Since $\{D(r) : r \in R\}$ forms a basis, the stalk can be calculated as the limit

$$\widetilde{M}_{\mathfrak{p}} = \operatorname{colim}_{\mathfrak{p} \in D(r)} M_r.$$

For each $r \in R$ with $\mathfrak{p} \in D(r)$, there is a localization map $M_r \to M_{\mathfrak{p}}$, so the universal property of colimits implies the existence of a map $\widetilde{M}_{\mathfrak{p}} \to M_{\mathfrak{p}}$. The inverse of this map is given by

$$M_{\mathfrak{p}} \to \widetilde{M}_{\mathfrak{p}}, \quad m/r \mapsto (m/r, D(r)),$$

so it is an isomorphism.

Problem 3. Let $X = \operatorname{spec}(R)$ and $Y = \operatorname{spec}(S)$ be affine schemes. Prove that there is a bijective correspondence

$$\{\text{morphisms of schemes } X \to Y\} \longleftrightarrow \{\text{homomorphisms of rings } S \to R\}.$$

Show that this correspondence sends isomorphisms to isomorphisms.

Solution 3. Recall that $\mathcal{O}_X(X) = R$ and $\mathcal{O}_Y(Y) = S$. Suppose that $\varphi : S \to R$ is a ring homomorphism. We construct a map of schemes $(f, f^{\#})$ from φ as follows. For any $\mathfrak{p} \in \operatorname{spec}(R)$ we set $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$. The preimage of a prime ideal is prime, so $f : X \to Y$ makes sense as a map of sets. Moroever, for any ideal $\mathfrak{a}$ we have $f^{-1}(V(\mathfrak{a})) = V(\phi(\mathfrak{a}))$ and so the preimages of closed sets are closed and therefore f is continuous. For any $s \in S$, we have $f^{-1}(D(s)) = D(\varphi(s))$ and we define

$$f^{\#} : \mathcal{O}_Y(D(s)) = S_s \to S_{\varphi(s)} = \mathcal{O}_X(D(\varphi) = f_*\mathcal{O}_X(D(s))$$

to be the obvious localization map. If $U \subseteq Y$ is an open set, then we may choose $s_1, \ldots, s_r$ such that the collection $\{D(s_i)\}$ cover U. It follows that for $r_i = \varphi(s_i)$ the collection $\{D(r_i)\}$ covers $f^{-1}(U)$ and therefore we have a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{O}_Y(U) & \longrightarrow & \prod_i \mathcal{O}_Y(D(s_i)) & \longrightarrow & \prod_{i \neq j} \mathcal{O}_Y(D(s_i s_j)) \\
 & & \Big\downarrow{\scriptstyle \exists!} & & \Big\downarrow{\scriptstyle f^{\#}} & & \Big\downarrow{\scriptstyle f^{\#}} \\
0 & \longrightarrow & \mathcal{O}_X(f^{-1}(U)) & \longrightarrow & \prod_i \mathcal{O}_X(D(\varphi(r_i))) & \longrightarrow & \prod_{i \neq j} \mathcal{O}_X(D(r_i r_j))
\end{array}$$

where the rows are exact and the vertical maps are the obvious localizations. Hence there exists a unique ring homomorphism $f^{\#} : \mathcal{O}_Y(U) \to f_*\mathcal{O}_X(U)$ making the above diagram commute. By definition, this map is compatible with restriction and therefore $(f, f^{\#})$ defines a morphism of ringed spaces. Furthermore, for $\mathfrak{q} = f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$ the induced map on stalks is $S_{\mathfrak{q}} \mapsto R_{\mathfrak{p}}$

is the localization map induced by φ which sends $\mathfrak{q}$ to $\varphi(\mathfrak{q}) \subseteq \mathfrak{p}$ and thus $(f, f^{\#})$ is a morphism of locally ringed spaces ie. a morphism of schemes.

Conversely, given any morphism of schemes $f : X \to Y$, restricting $f^{\#}$ to global sections defines a ring homomorphism $f^{\#} : S \to R$. Its clear from the previous construction that if $(f, f^{\#})$ is constructed from φ, then $f^{\#} : S \to R$ is precisely φ. Consequently the map taking homomorphisms $S \to R$ to morphism of schemes $X \to Y$ is injective.

To prove surjectivity, we must show that if $g : X \to Y$ is a morphism of schemes and $f : X \to Y$ is the morphism of schemes constructed from the ring homomorphism $\varphi : S \to R$ as before for $\varphi = g^{\#}$, then $f = g$. To see this, we will work with stalks. Since g is a morphism of schemes, we know that it is a map of locally ringd spaces: for any prime ideal $\mathfrak{p}$ of R, the induced map of stalks $g^{\#} : S_{\mathfrak{q}} \to R_{\mathfrak{p}}$ sends $g(\mathfrak{p})$ into $\mathfrak{p}$. Localization gives us a commutative diagram:

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & R \\ \downarrow & & \downarrow \\ S_{g(\mathfrak{p})} & \xrightarrow{g^{\#}} & R_{\mathfrak{p}} \end{array}$$

Since $g^{\#}(g(\mathfrak{p})) \subseteq \mathfrak{p}$, and $g(\mathfrak{p})$ is maximal in $S_{g(\mathfrak{p})}$, we know that $(g^{\#})^{-1}(\mathfrak{p}) = g(\mathfrak{p})$. In turn, the commutativity of the above diagram shows $\varphi^{-1}(\mathfrak{p}) = g(\mathfrak{p})$. Thus f and g agree as maps of topological spaces. This means that $f_*\mathcal{O}_X = g_*\mathcal{O}_X$ and therefore $f^{\#} - g^{\#} : \mathcal{O}_Y \to g_*\mathcal{O}_X$ is a well-defined morphism of schemes. The above shows that it is zero on stalks, and therefore must be the zero morphism. Hence $f^{\#} = g^{\#}$ and we have our bijection.

An isomorphism of schemes $f : X \to Y$ is a morphism of schemes $(f, f^{\#})$ where $f : X \to Y$ is a homeomorphism of topological spaces and $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism of sheaves of rings. Thus by definition, if $f : X \to Y$ is an isomorphism, then the associated map $f^{\#} : S \to R$ is a ring isomorphism. Conversely, suppose that $\varphi : S \to R$ is an isomorphism of rings and let $f : X \to Y$ be the induced morphism of schemes. Then as a map of topological spaces $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$ has an inverse map $\mathfrak{q} \mapsto \varphi(\mathfrak{q})$, which are both continuous for the same reason outlined above. Hence $f : X \to Y$ is a homeomorphism. Furthermore, since localization is exact we know that $f^{\#} : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ will be an isomorphism for all $U = D(s)$ and hence for all open $U \subseteq X$. Thus f is an isomorphism of schemes.

Problem 4. Let X be a scheme and for all $p \in X$ let $\mathfrak{m}_p$ denote the maximal ideal of the local ring $(\mathcal{O}_X)_p$ (the stalk of the structure sheaf at p). For any $f \in \mathcal{O}_X(X)$ we let f_p denote the image of f under the natural map $\mathcal{O}_X(X) \to (\mathcal{O}_X)_p$ and define

$$X_f = \{p \in X : \; f_p \notin \mathfrak{m}_p\}.$$

(a) Show that X_f is an open subscheme of X

(b) If $X = \text{spec}(R)$ and $r \in \mathcal{O}_X(X) = R$, show that $X_r = D(r)$ and is affine (in fact it is isomorphic to $\text{spec}(A_r)$ as a scheme)

(c) Show that an open subset of an affine scheme is not necessarily affine (eg. $\text{spec}(\mathbb{C}[x,y])\backslash\{(x,y)\}$)

(d) Show that if Y is an affine scheme, then there exist global sections $s_1, \dots, s_r \in \mathcal{O}_Y(Y)$ with Y_{s_i} affine for all i and $s_1, \dots, s_r$ generates the unit ideal on $\mathcal{O}_Y(Y)$.

BONUS: Prove that (d) is actually an if and only if condition.

Solution 4.

(a) Let $U_i = \text{spec}(A_i)$ be an affine open covering of X. Then for all i, let $f_i = \text{res}_{X,U_i}(f) \in \mathcal{O}_X(\text{spec}(A_i)) = A_i$ and note

$$\begin{aligned} U_i \cap X_f &= \{p \in U_i : f_p \notin \mathfrak{m}_p\} \\ &= \{\mathfrak{p} \in \text{spec}(A_i) : \text{image of } f \text{ under } A \to A_\mathfrak{p} \text{ not in the maximal ideal}\} \\ &= \{\mathfrak{p} \in \text{spec}(A_i) : \text{image of } f \text{ under } A \to A_\mathfrak{p} \text{ not in } \mathfrak{p}\} \\ &= \{\mathfrak{p} \in \text{spec}(A_i) : f \notin \mathfrak{p}\} = D(f_i) \subseteq \text{spec}(A_i) \subseteq X. \end{aligned}$$

Thus $X_f = \bigcup_i (U_i \cap X_f) = \bigcup_i D(f_i)$ which is a union of open sets and therefore open.

(b) This is obvious from the calculation in (a)

(c) Let $X = \text{spec}(\mathbb{C}[x,y])$ and $U = X\backslash\{\mathfrak{m}\}$ where $\mathfrak{m} = (x,y) \subseteq \mathbb{C}(x,y)$. Since $\mathfrak{m}$ is a maximal ideal in $\mathbb{C}[x,y]$, it is a closed point of X, and therefore U is open. Recall that

$$\mathcal{O}_X(U) = \{f(x,y)/g(x,y) : f,g \in \mathbb{C}[x,y], \; D(g) \subseteq U\}.$$

However, if $D(g) \subseteq U$ then $V(g) \subseteq \{\mathfrak{m}\}$ so the set $\{(a,b) \in \mathbb{C}^2 : g(a,b) = 0\}$ consists of only the point $(0,0)$. This implies that g is constant and therefore $\mathcal{O}_X(U) = \mathbb{C}[x,y]$.

If U is affine, ie. $U = \mathrm{spec}(A)$, then A must be $\mathcal{O}_X(U)$ and by the previous problem the inclusion $U \to X$ corresponds to a ring homomorphism $\mathbb{C}[x,y] \to \mathcal{O}_X(U)$. However, $A = \mathrm{spec}(\mathbb{C}[x,y])$ and the ring homomorphsim $\mathbb{C}[x,y] \to \mathcal{O}_X(U)$ is the identity. This means that $U \to X$ is an isomorphism of schemes, which is obviously false since it's not even a bijection as sets. Hence U is a scheme which is not affine.

(d) This is easy. If $Y = \mathrm{spec}(A)$, then the identity $1 \in A = \mathcal{O}_Y(Y)$ is a global section of Y and $Y_1 = Y$ is affine and 1 generates the unit ideal in A. The setup here was to make the bonus make sense, ie. so that the converse statement also holds.

BONUS: Suppose that Y is a scheme and that there exist global sections $s_1, \ldots, s_r$ of Y with Y_{s_i} affine for all i and with $s_1, \ldots, s_r$ generating the unit ideal in $\mathcal{O}_Y(Y)$. For each i, let $A_i = \mathcal{O}_Y(Y_i)$ and let $A = \mathcal{O}_Y(Y)$ and let $s_{ij} \in A_j$ be the image of s_i under the restriction map to Y_{s_j}. By the same calculation as in (a), we know that $Y_{s_i} \cap Y_{s_j} = Y_{s_i s_j}$ is affine and isomorphic to $\mathrm{spec}(A_{ij})$, where $A_{ij} = (A_i)_{s_{ji}} = (A_j)_{s_{ij}}$. Then we have an exact sequence of morphisms of schemes

$$0 \to Y \to \amalg_i Y_{s_i} \to \amalg_{i \neq j} Y_{s_i s_j}.$$

In terms of rings, we have an exact sequence of rings

$$0 \to A \to \prod_i A_i \xrightarrow{(a_i) \mapsto (a_i - a_j)} \prod_{i \neq j} A_{ij}.$$

However, this leads to a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & Y & \longrightarrow & \amalg_i Y_{s_i} & \longrightarrow & \amalg_{i \neq j} Y_{s_i s_j} \\
\| & & \| & & \downarrow & & \| & & \| \\
0 & \longrightarrow & 0 & \longrightarrow & \mathrm{spec}(A) & \longrightarrow & \amalg_i \mathrm{spec}(A_i) & \longrightarrow & \amalg_{i \neq j} \mathrm{spec}(A_{ij})
\end{array}$$

where each of the rows is exact. By the five lemma, it follows that Y is isomorphic to $\mathrm{spec}(A)$ and in particular is affine.