Algebraic Geometry Fall 2018 Homework 4

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Due Wednesday October 3 (start of class)

Problem 1. Let k be an algebraically closed field. Recall that a projective variety is an irreducible, closed subset of $\mathbb{P}(V)$ for some k-vector space V, where $\mathbb{P}(V) = \{[\vec{v}] : \vec{v} \in V\}$. The expression $[\vec{v}]$ denotes the equivalence class of $\vec{v}$, where two vectors are equivalent if and only if they are linearly dependent.

The grassmannian $G(m, n)$ is the collection of all m dimensional subspaces of k^n. It has the structure of a projective variety, via the Plücker embedding

$$G(m, n) \mapsto \mathbb{P}(\wedge^m k^n), \quad \text{span}\{\vec{v}_1, \dots, \vec{v}_m\} \mapsto [\vec{v}_1 \wedge \cdots \wedge \vec{v}_m] \in \mathbb{P}(\wedge^m k^n).$$

This map is well-defined and injective, so it identifies $G(m, n)$ with a certain subset of $\mathbb{P}(\wedge^m k^n)$. To prove that $G(m, n)$ is a projective variety, one shows that this image under the Plücker embedding is the zero set of a homogeneous prime ideal in the symmetric algebra $S(\wedge^m k^m)$ (or if we choose a specific basis, a polynomial ring of appropriate size).

(a) Show that the Plücker embedding is well-defined and injective.

(b) Let $\alpha \in \wedge^m k^n$. Show that $[\alpha]$ lies in the image of the Plücker embedding if and only if the kernel of the map $k^n \mapsto \wedge^{m+1} k^n : \vec{v} \mapsto \vec{v} \wedge \alpha$ has dimension m.

(c) Consider the first interesting case $G(2, 4)$. The dimension of $\wedge^2(k^4)$ is 6, with basis $\{\vec{e}_i \wedge \vec{e}_j : i < j\}$ by which we may identify it with k^6. Algebraic subsets of $\mathbb{P}(\wedge^2(k^4))$ may then be identified with zero sets of collections of homogeneous polynomials in $S = k[x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}]$. Show that the image of $G(2, 4)$ under the Plücker embedding is the zero set of a single irreducible, homogeneous polynomial of degree 2, and therefore in particular is a projective variety. This polynomial is called the Plücker relation for $G(2, 4)$. What is the dimension of $G(2, 4)$ as a variety? [Hint: try using the characterization of the image from (b)]

(d) (Challenging problem, not required) Can you figure out the Plücker relations in general? Can you think of a reason why the image of the Plücker embedding might be irreducible in general?

Problem 2. Let A be a graded k-algebra with A_0 reduced.

(a) Show the ring of global sections of the structure sheaf of $\mathrm{Proj}(A)$ contains A_0

(b) If A is an integral domain, prove $\mathrm{Proj}(A)$ is reduced and irreducible.

(c) Let B be another graded k-algebra and $\varphi : A \to B$ be a graded k-algebra homomorphism. Show that

$$V := \{\mathfrak{p} \in \mathrm{Proj}(B) : \varphi(A_+) \not\subseteq \mathfrak{p}\}$$

is an open subset of $\mathrm{Proj}(B)$ and that φ induces a morphism of schemes $V \to \mathrm{Proj}(A)$.

(d) Show by example that in general a graded k-algebra homomorphism $\varphi : A \to B$ may not induce a morphism of schemes $\mathrm{Proj}(B) \to \mathrm{Proj}(A)$, ie. that the homomorphism $V \to \mathrm{Proj}(A)$ from (c) may not extend.

Problem 3. Let A be a graded k-algebra, $X = \mathrm{Proj}(A)$, and M a graded A-module.

(a) Show that there exists a unique $\mathcal{O}_X$-module $\widetilde{M}$ on X satisfying $\widetilde{M}(D(f)) = (M_f)_0$ for all homogeneous $f \in A_+$, where here $(M_f)_0$ is the degree 0 component of the graded module M_f.

(b) For any integer ℓ, define a graded A-module $\mathrm{tail}_\ell(M) := \bigoplus_{n>\ell} M_n$. Show that $\widetilde{M} \cong \widetilde{\mathrm{tail}_\ell(M)}$ for all ℓ.
Remark: This means that passing from graded A-modules to $\mathcal{O}_X$-modules only remembers the tails of modules. In fact $M \mapsto \widetilde{M}$ defines an equivalence of categories between the category of tails of graded A-modules and the category of $\mathcal{O}_X$-modules.

Problem 4. Let A be a graded k-algebra, $X = \mathrm{Proj}(A)$, and M a graded A-module. The ℓ'th Serre twist of $\widetilde{M}$ is the module $\widetilde{M}(\ell) = \widetilde{M(\ell)}$, where $M(\ell)$ is the graded A module whose n'th homogeneous component is $M_{\ell+n}$ for all $n \geq 0$. In the special case $M = A$, we write $\mathcal{O}_X(\ell)$ in place of $\widetilde{A(\ell)}$. Show that for any ℓ the stalks $\mathcal{O}_X(\ell)_x$ at any $x \in X$ is free of rank 1 (ie. that $\mathcal{O}_X(\ell)$ is locally free of rank 1).
Remark: The collection of (isomorphism classes of) all locally free, rank 1 projective $\mathcal{O}_X$-modules forms an important object of study called the Picard group of X.