Algebraic Geometry Fall 2018 Homework 4

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Due Wednesday October 3 (start of class)

Problem 1. Let k be an algebraically closed field. Recall that a projective variety is an irreducible, closed subset of $\mathbb{P}(V)$ for some k-vector space V, where $\mathbb{P}(V) = \{[\vec{v}] : \vec{v} \in V\}$. The expression $[\vec{v}]$ denotes the equivalence class of $\vec{v}$, where two vectors are equivalent if and only if they are linearly dependent.

The grassmannian $G(m, n)$ is the collection of all m dimensional subspaces of k^n. It has the structure of a projective variety, via the Plücker embedding

$$G(m, n) \mapsto \mathbb{P}(\wedge^m k^n), \quad \text{span}\{\vec{v}_1, \dots, \vec{v}_m\} \mapsto [\vec{v}_1 \wedge \cdots \wedge \vec{v}_m] \in \mathbb{P}(\wedge^m k^n).$$

This map is well-defined and injective, so it identifies $G(m, n)$ with a certain subset of $\mathbb{P}(\wedge^m k^n)$. To prove that $G(m, n)$ is a projective variety, one shows that this image under the Plücker embedding is the zero set of a homogeneous prime ideal in the symmetric algebra $S(\wedge^m k^m)$ (or if we choose a specific basis, a polynomial ring of appropriate size).

(a) Show that the Plücker embedding is well-defined and injective.

(b) Let $\alpha \in \wedge^m k^n$. Show that $[\alpha]$ lies in the image of the Plücker embedding if and only if the kernel of the map $k^n \mapsto \wedge^{m+1} k^n : \vec{v} \mapsto \vec{v} \wedge \alpha$ has dimension m.

(c) Consider the first interesting case $G(2, 4)$. The dimension of $\wedge^2(k^4)$ is 6, with basis $\{\vec{e}_i \wedge \vec{e}_j : i < j\}$ by which we may identify it with k^6. Algebraic subsets of $\mathbb{P}(\wedge^2(k^4))$ may then be identified with zero sets of collections of homogeneous polynomials in $S = k[x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}]$. Show that the image of $G(2, 4)$ under the Plücker embedding is the zero set of a single irreducible, homogeneous polynomial of degree 2, and therefore in particular is a projective variety. This polynomial is called the Plücker relation for $G(2, 4)$. What is the dimension of $G(2, 4)$ as a variety? [Hint: try using the characterization of the image from (b)]

(d) (Challenging problem, not required) Can you figure out the Plücker relations in general? Can you think of a reason why the image of the Plücker embedding might be irreducible in general?

Solution 1.

(a) If $V \in G(m,n)$ and $\vec{u}_1, \dots, \vec{u}_m$ and $\vec{v}_1, \dots, \vec{v}_m$ are two bases for V, then there exists a linear isomorphism $T : V \to V$ with $T\vec{u}_i = \vec{v}_i$ for all i. Direct calculation shows

$$\vec{u}_1 \wedge \cdots \wedge \vec{u}_m = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} T_{\sigma(i),i} \vec{v}_1 \wedge \cdots \wedge \vec{v}_m = \det(T) \vec{v}_1 \wedge \cdots \wedge \vec{v}_m,$$

and therefore $[\vec{u}_1 \wedge \cdots \wedge \vec{u}_m] = [\vec{v}_1 \wedge \cdots \wedge \vec{v}_m]$. Furthermore, if we complete $\vec{v}_1, \dots, \vec{v}_m$ to a basis $\vec{v}_1, \dots, \vec{v}_n$ of k^n and let T be the matrix whose columns are the $\vec{v}_j$'s, then by the same calculation as before

$$(\vec{v}_1 \wedge \cdots \wedge \vec{v}_m) \wedge (\vec{v}_{m+1} \wedge \cdots \wedge \vec{v}_n) = \det(T) \vec{e}_1 \wedge \cdots \wedge \vec{e}_n,$$

which is not zero. Therefore $\vec{v}_1 \wedge \cdots \wedge \vec{v}_m$ is not zero and this shows that the Plücker embedding is well-defined. Now if $W \in G(m,n)$ with basis $\vec{w}_1, \dots, \vec{w}_m$ and $[\vec{w}_1 \wedge \cdots \wedge \vec{w}_m] = [\vec{v}_1 \wedge \cdots \wedge \vec{v}_m]$ then there exists $0 \neq c \in k$ such that $c(\vec{w}_1 \wedge \cdots \wedge \dots \vec{w}_m = \vec{v}_1 \wedge \cdots \wedge \vec{v}_m$. It follows that for all i that $\vec{w}_i \wedge \vec{v}_1 \wedge \cdots \wedge \vec{v}_m = 0$ and therefore $\vec{w}_i, \vec{v}_1, \dots, \vec{v}_m$ are linearly independent for all i. Consequently $\vec{w}_i \in V$ for all i, so that $W \subseteq V$ and since W and V have the same dimension this means $W = V$. Thus the Plücker embedding is injective.

(b) Let $V \in G(m,n)$ and let $\vec{v}_1, \dots, \vec{v}_m$ be a basis for V, which we complete to a basis $\vec{v}_1, \dots, \vec{v}_n$ for k^n. Set $\alpha = \vec{v}_1 \wedge \cdots \wedge \vec{v}_m$. Then by the calculation in (a), we know that $\vec{v}_1 \wedge \cdots \wedge \vec{v}_n$ is nonzero and therefore $\vec{v}_j \wedge \alpha \neq 0$ for $j = m+1, \dots, n$. Therefore the nullity of $\vec{v} \mapsto \vec{v} \wedge \alpha$ is at most m. Since the kernel contains the linearly independent vectors $\vec{v}_1 \wedge \cdots \wedge \vec{v}_m$, the nullity is exactly m. Thus if α is in the image of the Plücker embedding then $\vec{v} \mapsto \vec{v} \wedge \alpha$ has dimension m.

To prove the converse, we require the following result: a collection of linearly independent vectors $\vec{u}_1, \dots, \vec{u}_\ell$ belongs to the kernel of $\vec{v} \mapsto \vec{v} \wedge \alpha = 0$ if and only if $\alpha = \beta \wedge \gamma$ for some $\gamma \in \wedge^{m-\ell} k^n$, with $\beta = \vec{u}_1 \wedge \cdots \wedge \vec{u}_\ell$. To see this, expand $\vec{u}_1, \dots, \vec{u}_\ell$ to a a basis $\vec{u}_1, \dots, \vec{u}_n$ of k^n andconsider the expansion of α in terms of the canonical basis of $\wedge^m k^n$ induced by this basis:

$$\alpha = \sum_{i_1 < i_2 < \cdots < i_m} c_{i_1, i_2, \dots, i_m} \vec{u}_{i_1} \wedge \cdots \wedge \vec{u}_{i_m}.$$

Then since $\vec{v}_1 \wedge \alpha = 0$, we have

$$0 = \sum_{1 < i_1 < i_2 < \cdots < i_m} c_{i_1, i_2, \dots, i_m} \vec{v}_1 \wedge \cdots \wedge \vec{v}_\ell \wedge \vec{v}_{i_1} \wedge \cdots \wedge \vec{v}_{i_m}.$$

The entries $\vec{v}_1 \wedge \vec{v}_{i_1} \wedge \cdots \wedge_{i_m}$ are linearly independent for all $\ell < i_1 < \cdots < i_m$ so this implies the associated coefficients are zero. Hence

$$\alpha = \vec{u}_1 \wedge \sum_{1<i_2<\cdots<i_m} c_{1,i_2,\ldots,i_m} \vec{v}_{i_2} \wedge \cdots \wedge \vec{v}_{i_m}.$$

Next, wedging with $\vec{u}_2$ we see that $c_{1,i_2,\ldots,i_m} = 0$ for $i_2 \neq 2$. Continuing in this way, we find

$$\alpha = \vec{u}_1 \wedge \cdots \wedge \vec{u}_\ell \wedge \sum_{\ell<i_{\ell+1}<\cdots<i_m} \vec{v}_{1,\ldots,\ell,i_{\ell+1},\ldots,i_m} \vec{v}_{i_{\ell+1}} \wedge \cdots \wedge i_m.$$

This proves our claim.

Now if $\alpha \in \wedge^m k^n$ has an m dimensional kernel with basis $\vec{u}_1, \ldots, \vec{u}_m$, then the result of the previous paragraph tells us that $\alpha = \vec{u}_1 \wedge \cdots \wedge u_m \wedge \gamma \in \wedge^0 k^n = k$. Thus $[\alpha] = [\vec{u}_1 \wedge \cdots \wedge \vec{u}_m]$ is in the image of the Plücker embedding. This proves proves the converse.

(c) We identify

$$[x_{12} : x_{13} : x_{14} : x_{23} : x_{24} : x_{34}] \leftrightarrow [\alpha(x_{ij})] := \left[\sum_{i<j} x_{ij} \vec{e}_i \wedge \vec{e}_j\right] \in \mathbb{P}(\wedge^2 k^4).$$

Then the linear map $\vec{v} \mapsto \vec{v} \wedge \alpha(x_{ij})$ associated to the point with coordinates x_{ij} is

$$A(x_{ij}) := \begin{pmatrix} x_{23} & -x_{13} & x_{12} & 0 \\ x_{24} & -x_{14} & 0 & x_{12} \\ x_{34} & 0 & -x_{14} & x_{13} \\ 0 & x_{34} & -x_{24} & x_{23} \end{pmatrix},$$

expressed in terms of the standard basis $\vec{e}_{ijk}$ of $\wedge^3 k^4$ with the usual lexicographical order. The determinant of this matrix is given by

$$\det(A(x_{ij})) = (x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23})^2.$$

and its cofactor matrix is given by

$$\begin{aligned} \operatorname{cof}(A(x_{ij})) &= \det(A(x_{ij}))A(x_{ij})^{-1} \\ &= \begin{pmatrix} x_{14} & -x_{13} & x_{12} & 0 \\ x_{24} & -x_{23} & 0 & x_{12} \\ x_{34} & 0 & -x_{23} & x_{13} \\ 0 & x_{34} & -x_{24} & x_{14} \end{pmatrix} (x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}). \end{aligned}$$

From this we see rank of $A(x_{ij})$ is either $0, 2$, or 4, and the rank is 2 precisely when at least one of the x_{ij}'s is nonzero and the determinant vanishes. Thus the image of the Plücker embedding is exactly the projective variety

$$V(x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}) \subseteq \mathbb{P}(\wedge^2 k^4).$$

Problem 2. Let A be a graded k-algebra with A_0 reduced.

(a) Show the ring of global sections of the structure sheaf of $\mathrm{Proj}(A)$ contains A_0

(b) If A is an integral domain, prove $\mathrm{Proj}(A)$ is reduced and irreducible.

(c) Let B be another graded k-algebra and $\varphi : A \to B$ be a graded k-algebra homomorphism. Show that

$$V := \{\mathfrak{p} \in \mathrm{Proj}(B) : \varphi(A_+) \nsubseteq \mathfrak{p}\}$$

is an open subset of $\mathrm{Proj}(B)$ and that φ induces a morphism of schemes $V \to \mathrm{Proj}(A)$.

(d) Show by example that in general a graded k-algebra homomorphism $\varphi : A \to B$ may not induce a morphism of schemes $\mathrm{Proj}(B) \to \mathrm{Proj}(A)$, ie. that the homomorphism $V \to \mathrm{Proj}(A)$ from (c) may not extend.

Solution 2.

(a) Let $X = \mathrm{Proj}(A)$. Then X is covered by a collection of distinguished opens $\{D(f_i)\}$ corresponding to homogeneous elements $f_i \in A_+$. The global sections of X are precisely the kernel of the equalizer diagram

$$\mathcal{O}_X(X) = \ker\left(\prod_i \mathcal{O}_X(D(f_i)) \to \prod_{ij} \mathcal{O}_X(D(f_i f_j))\right).$$

By definition, this is precisely

$$\mathcal{O}_X(X) = \ker\left(\prod_i (A_{f_i})_0 \to \prod_{ij} (A_{f_i f_j})_0\right).$$

For $a \in A_0$ the section $(s_i) \in \prod_i (A_{f_i})_0$ with $s_i = a$ for all i lies in the kernel of the above, and therefore corresponds to an elment of $\mathcal{O}_X(X)$. Thus we have a map $A_0 \to \mathcal{O}_X(X)$ defined by sending $a \mapsto (s_i)$ with $s_i = a$ for all i. Since A is reduced, a cannot be nilpotent and the above map is injective.

(b) The fact that V is open follows from the definition of the topology of the Proj construction, since $V = \text{Proj}(B) \backslash V(\varphi(A_+))$. To define the induced map, it suffices to define it on affine opens and show things glue.

For $a \in A_+$ homogeneous and $b = \varphi(a)$ note that $D(b) \subseteq V$ because $\mathfrak{p} \in D(b)$ implies $\varphi(a) \notin \mathfrak{p}$ and therefore $\varphi(A_+) \nsubseteq D(g)$. Moreover, we have canonical isomorphisms $D(b) \cong \text{spec}((B_b)_0)$ and $D(a) \cong \text{spec}((A_0)_0)$ induced by the identity on global sections. For each $a \in A_+$ homogeneous we define $f_a : D(b) \to D(a)$ to be the morphism of affine schemes corresponding to the localized map $(A_a)_0 \to (B_b)_0$. For $a, \widetilde{a} \in A_+$ with $b = \varphi(a), \widetilde{b} = \varphi(b)$ we have a commutative diagram

$$\begin{array}{ccccccc}
 & & D(\widetilde{b}) & \longrightarrow & & & D(\widetilde{a}) \\
 & \nearrow^{\subseteq} & & & & \nearrow^{\subseteq} & \\
D(b\widetilde{b}) & & \longrightarrow & & D(a\widetilde{a}) & & \\
 & \searrow_{\subseteq} & & & & \searrow_{\subseteq} & \\
 & & D(b) & \longrightarrow & & & D(a)
\end{array}$$

This gives us a commutative diagram of morphisms of schemes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V & \longrightarrow & \coprod_{a \in A_+, homog.} D(\varphi(a)) & \rightrightarrows & \coprod_{a, \widetilde{a} \in A_+, \neq, homog.} D(\varphi(aa_+)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{Proj}(A) & \longrightarrow & \coprod_{a \in A_+, homog.} D(a) & \rightrightarrows & \coprod_{a, \widetilde{a} \in A_+, \neq, homog.} D(aa_+) & \longrightarrow & 0
\end{array}$$

where the rows are exact and come from the proj construction. The existence of the dashed arrow follows immediately from exactness of the rows and an obvious "diagram chase".

(c) Let $A = \mathbb{C}[x, y]$ and $B = \mathbb{C}[x, y, z]$ with the usual gradings. The inclusion $A \to B$ of graded rings does not produce a map $\mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$, because all such maps are constant! Indeed $\varphi(A_+)$ is contained in the homogeneous prime ideal (x, y) which is in $\mathrm{Proj}(B)$ and

$$V = \{\mathfrak{p} \in \mathrm{Proj}(B) : \varphi(A_+) \not\subseteq \mathfrak{p}\} = \mathbb{P}^2_{\mathbb{C}} \backslash \{[0:0:1]\}.$$

and the induced map $V \to \mathbb{P}^1_{\mathbb{C}}$ is just projection onto the first two coordinates $[a : b : c] \mapsto [a : b]$, which is undefined at the removed point $[0 : 0 : 1]$ of $\mathbb{P}^2_{\mathbb{C}}$.

Problem 3. Let A be a graded k-algebra, $X = \mathrm{Proj}(A)$, and M a graded A-module.

(a) Show that there exists a unique $\mathcal{O}_X$-module $\widetilde{M}$ on X satisfying $\widetilde{M}(D(f)) = (M_f)_0$ for all homogeneous $f \in A_+$, where here $(M_f)_0$ is the degree 0 component of the graded module M_f.

(b) For any integer ℓ, define a graded A-module $\mathrm{tail}_\ell(M) := \bigoplus_{n>\ell} M_n$. Show that $\widetilde{M} \cong \widetilde{\mathrm{tail}_\ell(M)}$ for all ℓ.
Remark: This means that passing from graded A-modules to $\mathcal{O}_X$-modules only remembers the tails of modules. In fact $M \mapsto \widetilde{M}$ defines an equivalence of categories between the category of tails of graded A-modules and the category of $\mathcal{O}_X$-modules.

Solution 3.

(a) For each homogeneous $a \in A_+$ let $\iota_a : D(a) \to X$ be the inclusion map. Recall that $D(a) \cong \mathrm{spec}((A_a)_0)$ so that the $(A_a)_0$-module $(M_a)_0$ gives rise to a sheaf $\mathcal{F}_a$ on $D(a)$ satisfying $\mathcal{F}_a(D(f)) = ((M_a)_0)_f$ for all $f \in (A_a)_0$. Furthermore for any homogeneous $b \in A_+$ we know that the natural restriction map $D(ab) \to D(a) \cap D(b)$ is an isomorphism (even if $a = b$) and this restriction induces an isomorphism of sheaves $\mathcal{F}_a|_{D(a)\cap D(b)} \cong \mathcal{F}_{ab}$ because the localizations of M are the same.

Let S be the set of all homogeneous elements of A_+ and as an abuse of notation write $s|_V$ for $\mathrm{res}_{U,V}(s)$ for $V \subseteq U \subseteq X$ and s a section over U of a sheaf. Consider the morphism of $\mathcal{O}_X$-modules on X

$$\prod_{a\in S} \iota_{a*}(\mathcal{F}_a) \to \prod_{a,\tilde{a}\in S} \iota_{a\tilde{a}*}(\mathcal{F}_{a\tilde{a}}).$$

defined for each $U \subseteq X$ by restriction:

$$\prod_{a \in S} \mathcal{F}_a(U \cap D(a)) \to \prod_{a, \widetilde{a} \in S} \mathcal{F}_{a\widetilde{a}}(U \cap D(a\widetilde{a}))$$
$$(s_a)_{a \in S} \mapsto ((s_a)|_{U \cap D(a\widetilde{a})} - (s_{\widetilde{a}})|_{U \cap D(a\widetilde{a})})_{a, \widetilde{a} \in S}$$

Let $\widetilde{M}$ denote the kernel of is morphism. Since kernels of morphisms of sheaves are automotically sheaves, we know $\widetilde{M}$ is a sheaf. Moreover, since the above is an $\mathcal{O}_X$-module homomorphism we know $\widetilde{M}$ is a $\mathcal{O}_X$-module.

For any homogeneous $b \in A_+$ we have

$$\iota_{a*}\mathcal{F}_a(D(b)) = \mathcal{F}_a(D(b) \cap D(a)) = M_{ab} = \mathcal{F}_b(D(b) \cap D(a))$$

so that we have an exact sequence

$$0 \to \mathcal{M}(D(b)) \to \prod_{a \in S} \mathcal{F}_b(D(b) \cap D(a)) \to \prod_{a, \widetilde{a} \in S} \mathcal{F}_b(D(b) \cap D(a) \cap D(\widetilde{a}))$$

where the arrows are defined by restriction. Since $\mathcal{F}_b$ is a sheaf on $D(b)$ and $\{D(a) \cap D(b) : a \in S\}$ is a cover of $D(b)$, the gluing property automatically tells us $\mathcal{M}(D(b)) = \mathcal{F}_b(D(b)) = (M_b)_0$.

(b) Fix $\ell \geq 0$ and set $N = \mathrm{tail}_\ell(M)$ and consider the short exact sequence of graded A-modules

$$0 \to N \overset{\subseteq}{\to} M \to T \to 0$$

where $T = M/N$. Note that the degree d'th component T_d of T is zero for $d \gg 0$ and therefore if $t \in T$ and $a \in A_+$ is homogeneous then $t/a^n = a^d t/a^{n+d} = 0$ so that the localization T_a is zero for all a.

The map from modules to sheaves is a functor and therefore sends the sequence to a sequence of modules

$$0 \to \widetilde{N} \to \widetilde{M} \to \widetilde{T} \to 0$$

which is also exact (as may be checked on stalks). Furthermore for every homogeneous $a \in A_+$ we have $\widetilde{T}(D(a)) = (T_a)_0$ by (a) and since $T_a = 0$ for all a, it follows that $\widetilde{T}$ is locally zero. By uniqueness of the gluing property of sheaves, it follows that $\widetilde{T} = 0$ and therefore $\widetilde{N}$ and $\widetilde{M}$ are isomorphic as sheaves.

Problem 4. Let A be a graded k-algebra, $X = \text{Proj}(A)$, and M a graded A-module. The ℓ'th Serre twist of $\widetilde{M}$ is the module $\widetilde{M}(\ell) = \widetilde{M(\ell)}$, where $M(\ell)$ is the graded A module whose n'th homogeneous component is $M_{\ell+n}$ for all $n \geq 0$. In the special case $M = A$, we write $\mathcal{O}_X(\ell)$ in place of $\widetilde{A(\ell)}$. Show that for any ℓ the stalks $\mathcal{O}_X(\ell)_x$ at any $x \in X$ is free of rank 1 (ie. that $\mathcal{O}_X(\ell)$ is locally free of rank 1).
Remark: The collection of (isomorphism classes of) all locally free, rank 1 projective $\mathcal{O}_X$-modules forms an important object of study called the Picard group of X.

Solution 4. For this question, we require an additional constraint in order for the statement to be true, avoiding certain silly situations where the grading has no control over the ring. Specifically we will assume that A is generated as an A_0-algebra by A_1. Note that many situations may be reduced to this case by taking a Veronese subring which doesn't change the value of $\text{Proj}(A)$, but which constrains our Serre shifts to be certain multiples of an integer.

Fix $\ell \in \mathbb{Z}$. For any nonzero $f \in A_1$, the map $(A(\ell)_f)_0 \to (A_f)_0$ defined by multiplication by $f^{-\ell}$ is an $(A_f)_0$-module homomorphism and therefore $\widetilde{A(\ell)}|_{D(f)}$ is isomorphic to $\mathcal{O}_{D(f)}$ as $\mathcal{O}_{D(f)}$-modules. In particular for any $x \in D(f)$, the stalk $A(\ell)_x$ is free of rank 1. Since the elements of degree 1 generate A, the set $\{D(f) : 0 \neq f \in A_1\}$ is a cover of X, so this proves the statement for all $x \in X$.