Algebraic Geometry Fall 2018 Homework 5

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Due Wednesday October 17 (start of class)

Problem 1. Let $f : X \to Y$ be a morphism of schemes. Prove that if f is finite then the preimage of any point $y \in Y$ is a finite set. This is interesting, since the behavior of the algebraic data (structure sheaves) is dictating the topological behavior.
Possible strategy:

(1) First reduce to the case $f : \text{spec}(A) \to \text{spec}(B)$ with A a finitely generated B module

(2) Next reduce to the case $f : \text{spec}(A) \to \text{spec}(B)$ with B an integral domain and y corresponding to the zero ideal of B and with A a finitely generated A module

(3) Next reduce to the case $f : \text{spec}(A) \to \text{spec}(B)$ with B a field and A a finite dimensional vector space

(4) Explain why A from (c) has finitely many prime ideals. (Lots of ways to do this, for example A must be Artinian and Noetherian)

Problem 2. Let $f : X \to Y$ be a morphism of schemes. Recall that f is qcqs means f is quasi-compact and quasi-separated.

(a) Prove that qcqs is an affine local property on the target space. In other words, prove that f is qcqs if and only if there exists an affine cover $Y = \bigcup_i V_i$ with $V_i = \text{spec}(B_i)$ such that $f^{-1}(V_i)$ is quasi-compact and quasi-separated for all i. Hint: use the affine communication lemma.

(b) Give an example of a qcqs morphism f

(c) Give an example of a morphism which is not qcqs.

Problem 3. In class, we proved that a morphism being affine is an affine local property on the target space. In our proof, we used the following fact (called the qcqs lemma): If X is qcqs, then the natural map

$$\mathcal{O}_X(X)_f \to \mathcal{O}_X(X_f)$$

is an isomorphism for every $f \in \mathcal{O}_X(X)$. Here X_f denotes the open set $X_f = \{x \in X : f_x \notin m_x \subseteq \mathcal{O}_{X,x}\}$.

(a) What is the natural map described above?

(b) Prove the qcqs lemma. The key observation is that localization respects *finite* products.

Problem 4. Let Y be a scheme. For each affine open set $\operatorname{spec}(B) \subseteq Y$ let $I(B)$ be an ideal of B. Show that if the natural map $I(B)_f \to B_f$ maps defines an isomorphism $I(B)_f \cong I(B_f)$ for all $f \in B$ then there exists a scheme X and a closed embedding $\pi : X \to Y$ such that the ideal sheaf of X over Y satisfies

$$\mathcal{I}_{X/Y}(\operatorname{spec}(B)) = I(B)\ \forall \operatorname{spec}(B) \subseteq Y.$$