# Algebraic Geometry Fall 2018 Homework 6

W.R. Casper

Due Wednesday October 24 (start of class)

**Problem 1.** Show that $\mathbb{P}^1_k \times \mathbb{P}^1_k$ is not isomorphic to $\mathbb{P}^2_k$ as a scheme. [Hint: there are many ways to see this. One way is to show that $\mathbb{P}^1_k \times \mathbb{P}^1_k$ has a non-constant map to $\mathbb{P}^1_k$, whereas every map from $\mathbb{P}^2_k$ to $\mathbb{P}^1_k$ is constant. Can you think of more reasons?]

**Problem 2.** A morphism $f : X \to Y$ is called **projective** if it factors as $X \xrightarrow{i} \mathbb{P}^n_Y \to Y$ with $i$ a closed embedding, where $\mathbb{P}^n_Y := \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} Y$ and $\mathbb{P}^n_Y \to Y$ is the canonical morphism. We also say $X$ is projective over $Y$.

(a) Prove that projective morphisms are stable under base change. [Hint: $X \times_{\mathbb{P}^n_Z} \mathbb{P}^n_Y \cong X \times_Z Y$]

(b) Prove that a composition of projective morphisms is projective. [Hint: $\mathbb{P}^m_{\mathbb{P}^n_Y} \cong \mathbb{P}^m_Y \times_Y \mathbb{P}^n_Y$ and embeds into $\mathbb{P}^{(m+1)(n+1)-1}_Y$ via the Segre embedding]

**Problem 3.** Let $h : X \to Y$ be a morphism of schemes over $Z$ (meaning that there are morphisms $f : X \to Z$, and $g : Y \to Z$ and the obvious triangle is a commutative diagram). The **graph morphims of** $h$ is the morphism $\Gamma_h : X \to X \times_Z Y$ induced by the universal property of fibered products. An important special case is the **diagonal morphism** $\Delta_{X/Z} := \Gamma_{\text{id}} : X \to X \times_Z X$ and $X$ is called **separated over** $Z$ (or equiv. $f$ is called a separated) if $\Delta_{X/Z}$ is a closed embedding.

(a) Show that if $Y$ is separated over $Z$ then $\Gamma_h$ is a closed embedding. [Hint: closed embeddings are preserved under base change]

(b) The morphism $f : X \to Z$ is called **closed** if it maps closed sets to closed sets and **universally closed** if every base change of $f$ is closed. Show that if $f$ is universally closed and $g$ is separated then $h$ is closed. [Hint: factor $h$ as $X \to X \times_Z Y \to Y$]