Algebraic Geometry Fall 2018 Homework 7

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Due Wednesday October 31 (start of class)

Problem 1. Let A be a ring and $n > 0$ an integer and let $S = A[x_0, \dots, x_n]$. In this problem, we explore the tautological morphism $\pi : \mathbb{P}^n_A \to \operatorname{spec}(A)$.

(a) Recall that the residue field of $\mathfrak{p} \in \operatorname{spec}(A)$ is $\kappa(\mathfrak{p}) := \operatorname{spec}(A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p})$. For each $\mathfrak{p} \in \operatorname{spec}(A)$ the obvious graded ring homomorphism $\varphi_\mathfrak{p} : S \to \kappa(\mathfrak{p})[x_0, \dots, x_n]$ defines a morphism $\iota_\mathfrak{p} : \mathbb{P}^n_{\kappa(\mathfrak{p})} \to \mathbb{P}^n_A$.

Prove that the following diagram commutes

$$\begin{array}{ccc} \mathbb{P}^n_{\kappa(\mathfrak{p})} & \longrightarrow & \operatorname{spec}(\kappa(\mathfrak{p})) \\ \downarrow{\scriptstyle \iota_\mathfrak{p}} & & \downarrow \\ \mathbb{P}^n_A & \longrightarrow & \operatorname{spec}(A) \end{array}$$

(b) Let $Z \subseteq \mathbb{P}^n_A$ be closed. By definition of $\mathbb{P}^n_A = \operatorname{Proj}(S)$, we know $Z = V(\{f_1, f_2, \dots\})$ for some collection of homogeneous polynomials $\{f_i\} \subseteq S$. Prove that the image of Z in $\operatorname{spec}(A)$ is

$$\pi(Z) = \{\mathfrak{p} \in \operatorname{spec}(A) : V(\{\varphi_\mathfrak{p}(f_1), \varphi_\mathfrak{p}(f_2), \dots\}) \subseteq \mathbb{P}^n_{\kappa(\mathfrak{p})} \text{ is nonempty}\}.$$

[Hint: use the diagram from (a)]

(c) As an interesting special case, suppose that $n = 2$, A is a polynomial ring in 9 variables $A = k[y_{00}, y_{01}, \dots, y_{22}]$ and

$$Z = V(\{f_0, f_1, f_2\}), \quad f_i(x_0, x_1, x_2) := \sum_{j=0}^{2} y_{ij} x_j.$$

Prove that

$$\pi(Z) = V(h) \subseteq \operatorname{spec}(A), \quad h = \det \begin{pmatrix} y_{00} & y_{01} & y_{02} \\ y_{10} & y_{11} & y_{12} \\ y_{20} & y_{21} & y_{22} \end{pmatrix}.$$

In particular $\pi(Z)$ is closed. [Hint: $V(\{\varphi_\mathfrak{p}(f_0), \varphi_\mathfrak{p}(f_1), \varphi_\mathfrak{p}(f_2)\}) \subseteq \mathbb{P}^2_{\kappa(\mathfrak{p})}$ is nonempty if and only if the ideal generated by $\{\varphi_\mathfrak{p}(f_0), \varphi_\mathfrak{p}(f_1), \varphi_\mathfrak{p}(f_2)\}$) does not contain the irrelevent ideal (x_0, x_1, x_2) of $\kappa(\mathfrak{p})[x_0, x_1, x_2]$]

Problem 2. Adopting the notation of the last problem, prove that the tautological morphism $\pi : \mathbb{P}^n_A \to \operatorname{spec}(A)$ is a closed map by following the steps outlined below.

Let $Z = V(\{f_1, f_2, \dots\})$ for homogeneous polynomials $f_1, f_2, \dots \in S$.

(a) Prove that

$$\pi(Z) = \{\mathfrak{p} \in \operatorname{spec}(A) : \forall N \geq 0,\ (S/\mathfrak{p})_N \not\subseteq (\varphi_{\mathfrak{p}}(f_1), \varphi_{\mathfrak{p}}(f_2), \dots) \subseteq \kappa(\mathfrak{p})[x_0, \dots, x_n]\}.$$

Here $(S/\mathfrak{p})_N$ is the subspace of homogeneous elements of degree N in $S/\mathfrak{p} = \kappa(\mathfrak{p})[x_0, \dots, x_n]$.

(b) Let $d_i = \deg(f_i)$. For each N, consider the A-linear map

$$L_N : \bigoplus_i S_{N-d_i} \mapsto S_N, \quad (g_1, g_2, \dots) \mapsto \sum_i f_i g_i.$$

Prove that

$$\pi(Z) = \{\mathfrak{p} \in \operatorname{spec}(A) : \forall N \geq 0,\ \varphi_{\mathfrak{p}}(\operatorname{img}(L_N)) \neq S_N\}.$$

(c) Since L_N is A-linear we may write it as a big $|S_N| \times m_N$ matrix for some m_N (possibly infinite). For each N let $\{h_{N,\,j} \in A : j = 1, 2, \dots\}$ be the collection of all possible $|S_N| \times |S_N|$ minors of this matrix. Prove that

$$\pi(Z) = V(\{h_{N,\,j} : N \geq 0,\ j \geq 1\}).$$

In particular the image of Z is closed and since Z was an arbitrary closed subset of $\mathbb{P}^n_A$ this shows π is a closed map.

Problem 3. Show by example that $\mathbb{A}^n_A \to \operatorname{spec}(A)$ does not need to be a closed map. (For example, you could consider the case $A = k[y]$ and $n = 1$.)

Problem 4. Prove that proper morphisms are

(a) local on the target

(b) stable under base change

(c) closed under composition