Algebraic Geometry Fall 2018 Homework 8

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Due Monday October 26 (start of class)

Problem 1. Let A be a ring and $n > 0$ an integer, $S = A[x_0, \dots, x_n]$, and $X = \text{Proj}(S)$. In this problem, we explore the cohomology of line bundles on X. Recall that up to isomorphism, the only line bundles on X are the Serre twists $\mathcal{O}_X(m)$, $m \in \mathbb{Z}$, so our task is to compute $H^p(X, \mathcal{O}(m))$ for all $p \geq 0$ and $m \in \mathbb{Z}$. Note immediately that since X is covered by $n+1$ affines, that the cohomology is zero for $p > n$, so we need only worry about values up to $p = n$.

(a) Consider the affine open cover $\{U_i\}_{i=0}^n$ of X where $U_i = D(x_i)$. Show that for any $I \subseteq \{0, 1, \dots, n\}$ there exists an A-module isomorphism

$$\mathcal{O}_X(m)(U_I) \cong \text{span}_A \left\{ x_0^{i_0} x_1^{i_1} \dots x_n^{i_n} : \sum_j i_j = m,\ i_j \geq 0\ \forall j \notin I \right\}$$

such that for $I \subseteq J$ the restriction map $\mathcal{O}_X(m)(U_I) \to \mathcal{O}_X(m)(U_J)$ corresponds to inclusion.

(b) Use Čech cohomology to show that

$$H^0(X, \mathcal{O}(m)) = \text{span}_A \left\{ x_0^{i_0} x_1^{i_1} \dots x_n^{i_n} : \sum_j i_j = m,\ i_j \geq 0 \forall j \right\}$$

and also that

$$\begin{aligned} &H^n(X, \mathcal{O}(m)) \\ =&\text{span}_A \left\{ x_0^{i_0} x_1^{i_1} \dots x_n^{i_n} : \sum_j i_j = m \right\} / \text{span}_A \left\{ x_0^{i_0} x_1^{i_1} \dots x_n^{i_n} : \sum_j i_j = m,\ \exists j\ i_j \geq 0 \right\} \\ \cong&\text{span}_A \left\{ x_0^{i_0} x_1^{i_1} \dots x_n^{i_n} : \sum_j i_j = m,\ i_j \leq -1 \forall j \right\}. \end{aligned}$$

(c) From (b), we know that $H^0(X, \mathcal{O}(m))$ and $H^n(X, \mathcal{O}(m))$ are free A-modules. What are their ranks?

Problem 2. Continuing with the previous problem, in this problem we will show that the cohomology groups that we calculated above are in fact the only nonzero ones. The trick is to consider the rest all at once by defining $\mathcal{F} := \sum_{m \in \mathbb{Z}} \mathcal{O}(m)$ which in particular has the property that $\mathcal{F}(-1) = \mathcal{F}$.

(a) Use the short exact sequence of graded S-module homomorphisms

$$0 \to S(-1) \xrightarrow{x_n \cdot} S \to S/x_n \to 0$$

to show that there exists a long exact sequence in cohomology

$$\cdots \to H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}) \to H^p(Y, f^*\mathcal{F}) \to \dots$$

where here $Y = \operatorname{Proj}(S/x_n)$ and $f : Y \to X$ is the closed embedding.

(b) Show that the map $H^p(X, \mathcal{F}(-1)) \to H^p(X, \mathcal{F})$ is given by multiplication by x_n.

(c) Use induction on n to prove that $H^p(X, \mathcal{F}(-1)) \xrightarrow{x_n} H^p(X, \mathcal{F})$ is an isomorphism for all $0 < p < n$. [Note: the tougher part is showing this for $p = 0$ and $p = n$.]

(d) The previous result says multiplication by x_n is an automorphism of $H^p(X, \mathcal{F})$ for $0 < p < n$. Use this to conclude that $H^p(X, \mathcal{F}) = 0$ for $0 < p < n$.

Problem 3. Let X and Y be quasicompact and separated k-schemes and $\mathcal{F}, \mathcal{G}$ quasicoherent sheaves on X and Y, respectively. The **outer product** $\mathcal{F} \boxtimes \mathcal{G}$ of $\mathcal{F}$ and $\mathcal{G}$ is a sheaf on $X \times Y$ defined by

$$\mathcal{F} \boxtimes \mathcal{G} := \pi_X^* \mathcal{F} \otimes_{\mathcal{O}_{X \times_k Y}} \pi_Y^* \mathcal{G},$$

where here $\pi_X : X \times_k Y \to X$ and $\pi_Y : X \times_k Y \to Y$ are the standard projection maps.

1. Prove that $\mathcal{F} \boxtimes \mathcal{G}$ is quasicoherent.

2. Prove the Künneth formula

$$H^m(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) \cong \bigoplus_{p+q=m} H^p(X, \mathcal{F}) \otimes_k H^q(X, \mathcal{G}).$$

   [Hint: Use the fact that the cohomology of a tensor product of complexes is the tensor product of the cohomologies. Show that the tensor product of Čech complexes for $\mathcal{F}$ and $\mathcal{G}$ gives a Čech complex for their outer product.]

3. The line bundles on $\mathbb{P}^1_k \times \mathbb{P}^1_k$ are give by (up to isomorphism) $\mathcal{O}(m, n) := \mathcal{O}(m) \boxtimes \mathcal{O}(n)$. Use (b) to calculate $\dim_k H^p(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(m, n))$ for all m, n, p.