Group Schemes Homework 2

Due Monday, April 13

Problem 1. Prove that if $F : \underline{k\text{-Algs}} - \underline{\text{Sets}}$ is representable, then F satisfies the following properties:

S1 (uniqueness) for every k-algebra R and every finite family of k-algebras $\{R_i\}$ with $R \to \prod_i R_i$ faithfully flat

$$F(R) \to \prod_i F(R_i) \text{ is injective}$$

S2 (gluing) for every k-algebra R and every finite family of k-algebras $\{R_i\}$ with $R \to \prod_i R_i$ faithfully flat,

$$\text{if } \ (s_i) \in \prod_i F(R_i), \text{ and } s_i|_{R_i \to R_{ij}} = s_j|_{R_j \to R_{ij}} \ \forall i, j$$

$$\text{then } \ \exists s \in R_i \ \text{ with } \ s|_{R \to R_i} = s_i \ \forall i.$$

Here $R_{ij} = R_i \otimes_R R_j$ and for $s \in F_A$ and $A \xrightarrow{\varphi} B$, $s|_{A \xrightarrow{\varphi} B} := F(\varphi)(s)$.

Note that this is precisely the definition of F being a sheaf in the flat topology. In particular, representable functors are sheaves in the flat topology.

Problem 2. Consider the subgroups SL_n and $\mathbb{G}_m$ of GL_n. Then the functor $F : \underline{k\text{-Algs}} - \underline{\text{Sets}}$ defined by

$$F(R) = \mathrm{SL}_n(R)\mathbb{G}_m(R) \subseteq \mathrm{GL}_n(R)$$

is group-valued, but is not an affine group scheme because it is not representable.

Hint: let R be a ring with an element $r \in R$ which does not have an n'th root of unity, and let $R' = R[r^{1/n}]$ so that $R \to R'$ is faithfully flat. If R is representable, the previous problem says

$$\mathrm{img}(F(R) \to F(R')) = \{s \in F(R') : s|_{R' \xrightarrow{p_1} R' \otimes_R R'} = s|_{R' \xrightarrow{p_2} R' \otimes_R R'}\},$$

where here $p_i : R' \to R' \otimes_R R'$ is the map $r \mapsto r \otimes 1$ and $r \mapsto 1 \otimes r$, for $i = 1, 2$ respectively. Now show that this is *not* the case.

Problem 3. Let H and K be subgroups of an affine group G and let HK be the sheafification of the presheaf

$$R \mapsto H(R)K(R).$$

(a) Show that HK is representable b $\mathcal{O}_G/I$ where here I is the kernel of the homomorphism $\mathcal{O}_G \to \mathcal{O}_H \otimes_k \mathcal{O}_K$ defined by the obvious natural transformation $H \times K \to G$.

(b) Show that

$$(HK)(R) = \bigcup_{R \to R'} G(R) \cap (H(R')K(R'))$$

where the intersection is taken over all faithfully flat extensions $R \to R'$.

Problem 4. Use the previous result to show that $\mathrm{SL}_n \mathbb{G}_m = \mathrm{GL}_n$.