Course Notes

February 17, 2020

1 Day 1: Introduction and Category Theory

1.1 Introduction

Definition 1. A **topological group** G is a group which is a topological space with the property that the multiplication and inversion maps

$$G \times G \to G, \quad (g, h) \mapsto gh$$

$$G \to G, \quad g \mapsto g^{-1}$$

are continuous.

Some obvious examples include finite groups with the discrete topology, or matrix groups with the topologies inhereted from Euclidean space.

Just like with topology, we can study a collection of enriched topological spaces, ie. some collection of topological spaces with additional structure (smooth, algebraic), and we can consider topological groups wherein the given extra structure is respected by the group operations. This leads, for example, to the notion of Lie groups: topological groups with a smooth structure such that multiplication and inversion are smooth maps. The same principle leads us to groups schemes: topological groups which have a scheme structure, such that the multiplication and inversion maps are *algebraic*. Here, intuitively algebraic means "defined by rational functions".

Example 2. Let k be a field. The prototypical example of a group scheme is $\mathrm{GL}_N(k)$, the set of $N \times N$ invertible matrices with entries in k. The points of $\mathrm{GL}_N(k)$ may be identified with the (closed) points of an affine scheme

$$\mathrm{GL}_N(k) \leftrightarrow \mathrm{Spec}(\mathbb{C}[x_{11}, x_{12}, \ldots, x_{NN}, t]/(t\det(x_{ij}) - 1) \subseteq k^{N \times N+1}$$

by sending a matrix A with entries a_{ij} to the tuple $(a_{11}, a_{12}, \ldots, a_{NN}, \det(A)^{-1})$. The multiplication of matrices A and B with entries a_{ij} and b_{ij} satisfies

$$(AB)_{ik} = \sum_j a_{ij} b_{jk}.$$

Since it is defined by polynomials, it is algebraic. Likewise, Cramer's rule allows us to express the inverse of A as

$$A^{-1} = \operatorname{adj}(A) \det(A)^{-1},$$

where here $\operatorname{adj}(A)$ is the adjugate of A, whose entries are defined by polynomials in terms of the determinants of cofactors. Consequently inversion is defined by rational functions and is therefore also algebraic.

Example 3. A very different example of an algebraic group comes from algebraic geometry. Let g_1, g_2 be elements of a field k and consider the projective scheme

$$C = \operatorname{Proj}(k[X, Y, Z]/(Y^2Z - X^3 + g_1XZ^2 + g_2Z^3)).$$

The closed points of C correspond to equivalence classes $[a : b : c]$ with $a, b, c \in k^3$ (not all zero) satisfying $b^2c - a^3 + g_1ac^2 + g_2c^3$, where $[a : b : c] = [a' : b' : c']$ if (a, b, c) and (a', b', c') are linearly dependent. Thus as a set, the closed points of C may be identified with

$$\{\infty\} \cup \{(a, b) \in k^2 : b^2 = a^3 - g_1a - g_2\}.$$

where here ∞ is the "point at infinity" $\infty = [0 : 1 : 0]$ and (a, b) corresponds to $[a : b : 1]$.

As is well known, the curve C can be given an abelian group structure, where generically for any points $p = [a : b : 1]$ and $q = [a' : b' : 1]$, the value of $-(p+q)$ is the unique point R on C such that p, q, R are all colinear in the affine plane.

This can also be expressed in terms of line bundles. The distinguished point ∞ induces a map from points of C to the collection of line bundles $\mathcal{L}$ on C, sending a point p to the degree 0 line bundle $\mathcal{L}([p] - [\infty])$ with divisor class $[p] - [\infty]$. This is called the **Abel-Jacobi map**, which for an elliptic curve is a bijection with the set of line bundles. The sum of points p and q is the unique point $p+q$ satisfying the tensor product identity

$$\mathcal{L}([p] - [\infty]) \otimes \mathcal{L}([q] - [\infty]) \cong \mathcal{L}([p + q] - [\infty]).$$

An elliptic curve gives an example of a **projective group scheme**, ie. a group scheme where the underlying algebraic structure is a projective scheme (ie. has a closed embedding into projective space).

At least initially, we will focus our attention on **affine group schemes**, which are equivalent to commutative rings. Careful! This does not mean that the group operation will be abelian! For example, $\mathrm{GL}_N(k)$ above is not abelian, but is an affine group scheme.

Affine group schemes over a field k have three equivalent useful perspectives

- as representable functors $\underline{k\text{-Alg}} \to \underline{\text{Groups}}$
- as commutative Hopf algebras over k
- as groups in the category of schemes over k

We will explore each of these interpretations as we go along, starting with the very first. To do so, we need to review some category theory and in particular what it means for a functor to be representable.

1.2 Category Theory Review

We begin with a review of three important basic definitions from category theory.

Definition 4. A **category** $\mathcal{C}$ consists of a class $\text{obj}(\mathcal{C})$ of objects, a class $\text{hom}(\mathcal{C})$ of arrows (or morphisms, maps) such that associative composition and identities exist. As usual, $\text{hom}(A, B)$ denotes the class of arrows from A to B for all $A, B \in \text{obj}(\mathcal{C})$.

We will restrict our attention to locally small categories, wherein $\text{hom}(A, B)$ is a set for all $A, B \in \text{obj}(\mathcal{C})$. Natural examples of categories include the category $\underline{\text{Top}}$ of topological spaces and the category $\underline{\text{Set}}$ of sets.

Definition 5. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **functor** $F : \mathcal{C} \to \mathcal{D}$ is a mapping which sends each object $A \in \text{obj}(\mathcal{C})$ to an object $F(A) \in \text{obj}(\mathcal{D})$ and each arrow $f : A \to B$ to a morphism $F(f) : F(A) \to F(B)$ such that

$$F(\text{id}_A) = \text{id}_{F(A)} \quad \text{and} \quad F(g \circ f) = F(g) \circ F(f).$$

One example of a functor is $F : \underline{\text{Top}} \to \underline{\text{Set}}$ which takes a topological space X to the collection $F(X) = \{\gamma : S^1 \to X\}$ of continuous maps of S^1 into X, ie. loops in X. A homomorphism $f : X \to Y$ is sent to the set map $F(f) : F(X) \to F(Y)$, $\gamma \mapsto f \circ \gamma$.

Definition 6. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A **natural transformation** $T : F \to G$ is a collection of arrows $T_A : F(A) \to G(A)$ for all $A \in \text{obj}(\mathcal{C})$ such that for every morphism $f : A \to B$ in $\mathcal{C}$ we have

$$T_B \circ F(f) = G(f) \circ T_A.$$

If T_A is an isomorphism for all A, then T is called a **natural isomorphism** and F and G are called **isomorphic** and we write $F \cong G$.

2 Day 2: Representability

2.1 Yoneda's Lemma

One of the key observations of modern algebraic geometry is the following

Observation 7. An object A in a category $\mathcal{C}$ is determined (up to isomorphism) by its morphisms.

To begin, let's explore this idea from a topological perspective. Let X be a topological space. The collection $C(X)$ of continuous maps $X \to \mathbb{R}$ tells us a lot about X. In fact if X and Y are compact topological spaces, the Urysohn's Lemma implies that $C(X)$ and $C(Y)$ are isomorphic if and only if X and Y are homeomorphic. This can be viewed as an intuitive starting point for algebraic geometry: we can study the geometry of X by studying the structure of the algebra of continuous functions on X.

Inspired by the remarks of the previous paragraph, we ask how much information do we need to know about maps out of X do we need in order to figure out what X is (up to homeomorphism)? In general, knowing all maps $X \to \mathbb{R}$ is not enough (even in the compact case, unless we remember the algebra structure also). What the previous observation says is that if for any topological space Z, we know the set $C(X, Z)$ of continuous maps from X to Z, and we remember the natural maps $C(X, Z) \to C(X, Z')$ defined by homomorphisms $Z \to Z'$, then we can uniquely determine X.

To do things in an arbitrary category $\mathcal{C}$, we adopt the notation. For $A \in \mathrm{obj}(\mathcal{C})$ define the functor $h^A : \mathcal{C} \to \underline{\mathrm{Set}}$ by

$$\begin{aligned} h^A = \hom(A, -) :& B \mapsto \hom(A, B), \quad B \in \mathrm{obj}(A) \\ & f \mapsto (g \mapsto f \circ g, g \in \hom(A, B)), \quad f \in \hom(B, B') \end{aligned}$$

The functor h^A remembers the morphisms from A to other objects in $\mathcal{C}$, and what our starting observation amounts to is $h^A \cong h^{A'}$ if and only if $A \cong A'$. Furthermore, we can show that for any functor $F : \mathcal{C} \to \underline{\mathrm{Set}}$ there is a natural bijection between natural transformations $h^A \to F$ and objects in $F(A)$. This is the statement of Yoneda's Lemma.

Lemma 8 (Yoneda)**.** Let $\mathcal{C}$ be a category and let $F : \mathcal{A} \to \underline{\mathrm{Set}}$. There is a set bijection

$$\mathrm{Nat}(h^A, F) \cong F(A), \quad T \mapsto T_A(\mathrm{id}_A).$$

Moreover this bijection is natural in both A and F.

Proof. Fix an object A in $\mathcal{C}$. For any $a \in F(A)$, define a natural transformation $T_a : h^A \to F$ by

$$(T_a)_B : \hom(A, B) \to F(B), \quad f \mapsto F(f)(a).$$

Then for any natural transformation $T : h^A \to F$, the element $a = T_A(\mathrm{id}_A) \in F(A)$ satisfies

$$(T_a)_B(f) = F(f)(a) = F(f)(T_A(\mathrm{id}_A)) = T_B(h_A(f)(\mathrm{id}_A)) = T_B(f)$$

for all objects B in $\mathcal{C}$ and morphisms $f \in \hom(A, B)$.

Conversely, if $a \in F(A)$ then $(T_a)_A(\mathrm{id}_A) = F(\mathrm{id}_A)(a) = \mathrm{id}_{F(A)}(a) = a$. Thus $a \mapsto T_a$ and $T \mapsto T_A(\mathrm{id}_A)$ are inverse maps, showing that the map stated in the Lemma is a bijection. We leave the verification of the naturality of this bijection to the interested reader. □

Problem 1 (Due with Homework 1)**.** Use Yoneda's Lemma to prove that h^A and h^B are naturally isomorphic if and only if A and B are isomorphic.

2.2 Representability

Yoneda's lemma motivates the following definition.

Definition 9. A functor $F : \mathcal{C} \to \underline{\mathrm{Set}}$ is **representable** if there exists an object $A \in \mathrm{obj}(C)$ and a natural transformation $T : h^A \cong F$. In this case, we say that (A, T) **represents** F.

Without knowing any algebraic geometry, we can think of an affine scheme as a representable functor from $\underline{k\text{-Alg}}^{op}$ to $\underline{\text{Set}}$. In fact, the category of schemes is anti-equivalent to the category of k-algebras. In order to avoid having to build the theory of schemes, we will word directly with k-algebras and consider functors for $\underline{k\text{-Alg}}$ to $\underline{\text{Set}}$. We will also avoid arrows all over the place by defining affine schemes the following way:

Definition 10. An **affine scheme** is a representable functor $\underline{k\text{-Alg}} \to \text{Set}$.

Note that this is anti-equivalent to the usual category of affine schemes!

Example 11. The **affine line** $\mathbb{A}^1$ is the functor

$$\mathbb{A}^1 : \underline{k\text{-Alg}} \to \underline{\text{Set}}, \quad R \mapsto R.$$

This functor is representable by $A = k[x]$ since, we have a natural transformation

$$T : h^A \to \mathbb{A}^1, T_R : \hom(A, R) \to R, \ f \mapsto f(x)$$

which is an isomorphism.

Problem 2 (Due with Homework 1)**.** Show that there is no affine scheme F over k such that $G(R)$ has exactly two elements for all R.

3 Day 3: Affine Groups

In algebraic geometry, we learn that an affine scheme is controlled by its coordinate ring. Translating this into our format, we have the following definition.

Definition 12. Let $F : \underline{k\text{-alg}} \to \underline{\text{Set}}$ be an affine scheme. The **coordinate ring** of F is the set $\text{Nat}(F, \mathbb{A}^1)$.

Note that the coordinate ring of F has the structure of a k-algebra. The identity corresponds to the natural transformation

$$1_F : F \to \mathbb{A}^1, \quad (1_F)_R : F(R) \to R, \ a \mapsto 1.$$

Proposition 13. Let $F, \widetilde{F}$ be affine schemes and $A = \text{Nat}(F, \mathbb{A}^1)$ and $\widetilde{A} = \text{Nat}(\widetilde{F}, \mathbb{A}^1)$ be the corresponding coordinate rings. Then $F \cong \widetilde{F}$ if and only if $A \cong \widetilde{A}$. In particular, if $F \cong h^B$, then A and B are isomorphic.

Proof. Suppose that there exists an isomorphism $T : F \cong \widetilde{F}$. Then the natural map

$$\varphi : A \to \widetilde{A} : T' \mapsto T' \circ T$$

defines a k-algebra isomorphism.

Conversely, suppose that there exists a k-algebra isomorphism $\varphi : A \to \widetilde{A}$. Since F and $\widetilde{F}$ are representable, there exist natural transformations $T : h^B \cong F$ and $\widetilde{T} : h^{\widetilde{B}} \cong \widetilde{F}$ for some k-algebras $B, \widetilde{B}$. These induce algebra isomorphisms

$$B \overbrace{\cong}^{\text{Yoneda}} \text{Nat}(h^B, \mathbb{A}^1) \cong A \cong \widetilde{A} \cong \text{Nat}(h^{\widetilde{B}}, \mathbb{A}^1) \overbrace{\cong}^{\text{Yoneda}} \widetilde{B}.$$

Since B and $\widetilde{B}$ are isomorphic, it follows that h^B and $h^{\widetilde{B}}$ are isomorphic, and so F and $\widetilde{F}$ are isomorphic.

Note in particular that the this shows for $h^B \cong F$, we have

$$A \cong \mathrm{Nat}(h^B, \mathbb{A}^1) \overbrace{=}^{\text{Yoneda}} \mathbb{A}^1(B) = B$$

and thus A and B are isomorphic. $\square$

Corollary 14. Let $F, \widetilde{F}$ be affine schemes and $A = \mathrm{Nat}(F, \mathbb{A}^1)$ and $\widetilde{A} = \mathrm{Nat}(\widetilde{F}, \mathbb{A}^1)$ be the corresponding coordinate rings. Then

$$\mathrm{Nat}(F, \widetilde{F}) \cong \hom_k(\widetilde{A}, A).$$

Proof. Since $F \cong h^A$ and $\widetilde{F} \cong h^{\widetilde{A}}$ we hvae

$$\mathrm{Nat}(F, \widetilde{F}) \cong \mathrm{Nat}(h^A, h^{\widetilde{A}}) \overbrace{=}^{\text{Yoneda}} h^{\widetilde{A}}(A) = \hom_k(\widetilde{A}, A).$$

$\square$

3.1 Affine group

Now that we know what representability is, we can formally define an *affine group* over k.

Definition 15. A **affine group** over k is a representable functor $G : \underline{k\text{-Alg}} \to \underline{\mathrm{Set}}$ along with a natural transformation $m : G \times G \to G$ such that for all k-algebras R

$$m_R : G(R) \times G(R) \to G(R)$$

defines a group structure on $G(R)$. A **homomorphism** of **affine groups** $G \to H$ is a natural transformation which respects the group structure.

Example 16. The functor

$$\mathrm{GL}_N : \underline{k\text{-Alg}} \to \underline{\mathrm{Set}}, \quad R \mapsto \mathrm{GL}_N(R)$$

is representable by the k-algebra

$$A = k[x_{11}, x_{12}, \ldots, x_{NN}, y]/(y \det(x_{ij}) - 1).$$

To see this, note that there is a natural transformation

$$T : h^A \to \mathrm{GL}_N, \quad T_R : \hom(A, R) \to \mathrm{GL}_N(R),$$

defined by sending $\phi \in \hom(A, R)$ to $(\phi(x_{ij}))$. The natural transformation

$$m : \mathrm{GL}_N \times \mathrm{GL}_N \to \mathrm{GL}_N, \quad m_R : \mathrm{GL}_N(R) \times \mathrm{GL}_N(R) \to \mathrm{GL}_N(R)$$

defined by matrix products gives GL_N the structure of an affine group.

Example 17. The functor

$$\mathrm{SL}_N : \underline{k\text{-Alg}} \to \underline{\mathrm{Set}}, \quad R \mapsto \mathrm{SL}_N(R)$$

is a group scheme. In particular, it is representable by the k-algebra

$$A = k[x_{11}, x_{12}, \dots, x_{NN}, y]/(\det(x_{ij}) - 1).$$

Example 18. The functor

$$\mathbb{G}_a : \underline{k\text{-Alg}} \to \underline{\mathrm{Set}}, \quad R \mapsto R,$$

is representable by the k-algebra $k[x]$. Furthermore, it is a group scheme when we view R as a group with its additive group structure.

3.2 Products of Affine Groups

Let G and H be affine groups, with natural transformations

$$m_G : G \times G \to G, \quad m_H : H \times H \to H$$

defining group structures on $G(R)$ and $H(R)$ for all R. The product functor

$$G \times H : R \mapsto G(R) \times H(R).$$

can also be given a group structure by considering the natural transformation

$$m = m_G \times m_H : (G \times H) \times (G \times H) \to G \times H.$$

Furthermore, it is representable, making it an affine group.

Proposition 19. Let $F, \widetilde{F}$ be affine schemes. Then $F \times \widetilde{F}$ is also an affine scheme.

Proof. Let A and $\widetilde{A}$ be k-algebras with natural isomorphisms $h^A \cong T$ and $h^{\widetilde{A}} \cong \widetilde{F}$. Then for all k-algebras R, we have natural transformations

$$F(R) \times \widetilde{F}(R) \cong \hom_k(A, R) \times \hom_k(\widetilde{A}, R) \cong \hom_k(A \otimes_k \widetilde{A}, R).$$

Thus $F \times \widetilde{F}$ and $h^{A \otimes \widetilde{A}}$ are isomorphic. $\square$

4 Day 4: Hopf Algebras

4.1 Hopf Algebras

Definition 20. Let A be a k-algebra. A **co-multiplication** on an algebra A is a k-algebra homomorphism $\Delta : A \to A \otimes_k A$.

By Yoneda's lemma, this is the same as defining a natural transformation

$$h^{A\otimes_k A} = h^A \times h^A \to h^A,$$

ie. a binary operation on $h^A(R)$ for all R. If this defines a group structure on $h^A(R)$, then this makes h^A an affine group.

Conversely, if $G = h^A$ is an affine group, then $G \times G = h^{A\otimes_k A}$ and by Yoneda the multiplication operation $m : G \times G \to G$ corresponds to a co-multiplication $\Delta : A \to A \otimes_k A$.

To summarize: an affine group G is the same as a k-algebra A with a comultiplication Δ which *defines a group structure*. We want to explore further what it means for Δ to define a group structure. Our idea is now to translate the properties of a group into properties of the affine group scheme.

- The binary map defined by Δ is

  $$\begin{aligned} m : G(R) \times G(R) &\to G(R),\\ \hom_k(A,R)^{\oplus 2} &\to \hom_k(A,R), \quad (f,g) \mapsto \nabla \circ (f \otimes g) \circ \Delta. \end{aligned}$$

  where here ∇ is the algebra homomorphism $A \otimes_k A \to A$ defined in simple tensors by $\nabla(a \otimes b) = ab$.

- The existence of an identity in $G(R)$ for all R corresponds to a distinguished collection of k-algebra homomorphisms

  $$\epsilon_R : A \to R, \quad \text{such that} \quad \nabla \circ (\epsilon_R \otimes f) \circ \Delta = \nabla \circ (f \otimes \epsilon_R) \circ \Delta = f.$$

  Furthermore since m is a natural transformation, $\epsilon_R = e_R \circ \epsilon_k$ for $e_R : k \to R$ the unique k-algebra homomorphism. Thus ϵ_R is completely determined by $\epsilon = \epsilon_k$ satisfying

  $$(\epsilon \otimes \mathrm{id}_A) \circ \Delta = \mathrm{id}_A = (\mathrm{id}_A \otimes \epsilon) \circ \Delta.$$

- The existence of inverses corresponds to $S : h^A \to h^A$ such that

  $$S_R : \hom_k(A,R) \to \hom_k(A,R), \quad \nabla \circ (f \otimes S(f)) \circ \Delta = \nabla \circ (S(f) \otimes f) \circ \Delta = \epsilon_R.$$

  By Yoneda, this is the same thing as specifying an element of $S \in h^A(A) = \hom_A(A)$, which we call the **antipode** and which satisfies

  $$\nabla \circ (f \otimes f \circ S) \circ \Delta = \nabla \circ (f \circ S \otimes f) \circ \Delta.$$

- Associativity implies that for all $f, g, h \in \hom_k(A,R)$

  $$\nabla \circ ((\nabla \circ (f \otimes g) \circ \Delta) \otimes h) \circ \Delta = \nabla \circ (f \otimes (\nabla \circ (g \otimes h) \circ \Delta)) \circ \Delta,$$

  In the specific case when $R = A$ and $f = g = h = \mathrm{id}_A$, this says

  $$\nabla \circ ((\nabla \circ \Delta) \otimes \mathrm{id}_A) \circ \Delta = \nabla \circ (\mathrm{id}_A \otimes (\nabla \circ \Delta)) \circ \Delta,$$

or equivalently

$$\nabla \circ (\nabla \otimes \mathrm{id}_A) \circ (\Delta \otimes \mathrm{id}_A) \circ \Delta = \nabla \circ (\mathrm{id}_A \otimes \nabla) \circ (\mathrm{id}_A \otimes \Delta) \circ \Delta.$$

Noting that this morphism is injective and $\nabla \circ (\nabla \otimes \mathrm{id}_A) = \nabla \circ (\mathrm{id}_A \otimes \nabla)$, we obtain

$$(\Delta \otimes \mathrm{id}_A) \circ \Delta = (\mathrm{id}_A \otimes \Delta) \circ \Delta.$$

This is eqivalent to commutativity of

$$\begin{tikzcd}
& A \otimes_k A \arrow[dr, "\mathrm{id}_A \otimes \Delta"] & \\
A \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & A \otimes_k A \otimes_k A \\
& A \otimes_k A \arrow[ur, "\Delta \otimes \mathrm{id}_A"'] &
\end{tikzcd}$$

This motivates the definition of a coalgebra.

Definition 21. An **algebra** A consists of a tuple (A, ∇, e), with A a k-mudle, and linear maps $\nabla : A \otimes_k A \to A$ and $e : k \to A$ satisfying

$$\begin{aligned} &\text{unit} && \nabla \circ (e \otimes \mathrm{id}_A) = \mathrm{id}_A = \nabla \circ (\mathrm{id}_A \otimes \epsilon) && (1) \\ &\text{associativity} && \nabla \circ (\nabla \otimes \mathrm{id}_A) = \nabla \circ (\mathrm{id}_A \otimes \Delta). && (2) \end{aligned}$$

A homomorphism of algebras $(A, \nabla, e) \to (A', \nabla', e')$is a k-linear map $A \to A'$ making the obvious diagrams commute.

Definition 22. A **coalgebra** over k is a tuple (A, Δ, ϵ) consisting of a k-module A and k-linear maps $\Delta : A \to A \otimes_k A$ and $\epsilon : A \to k$ satisfying

$$\begin{aligned} &\text{counit} && (\epsilon \otimes \mathrm{id}_A) \circ \Delta = \mathrm{id}_A = (\mathrm{id}_A \otimes \epsilon) \circ \Delta && (3) \\ &\text{coassociativity} && (\Delta \otimes \mathrm{id}_A) \circ \Delta = (\mathrm{id}_A \otimes \Delta) \circ \Delta. && (4) \end{aligned}$$

A homomorphism of coalgebras $(A, \Delta, \epsilon) \to (A', \Delta', \epsilon')$is a k-linear map $A \to A'$ making the obvious diagrams commute.

Definition 23. A **bialgebra** over k is a quintuple $(A, \eta, e, \Delta, \epsilon)$ such that (A, η, e) is an algebra and (A, Δ, ϵ) is a coalgebra and Δ and ϵ are both algebra homomorphisms. A **inversion map** or **antipode** for a bialgebra is a k-linear map $S : A \to A$ such that the obvious things happen. A bialgebra with an antipode is called a **Hopf algebra**.

5 Day 9: More Examples of Group Schemes

5.1 More examples of basic group schemes

Example 24. The group μ_n defined by

$$\mu : \underline{k\text{-Alg}} \to \underline{\text{Set}}, \quad \mu_n(R) = \{r \in R : r^n = 1\}$$

is a group scheme with the obvious multiplication. It is represented by the k-algebra $k[x]/(x^n - 1)$.

Example 25. Let G_0 be a finite group and consider the set $A = \{f : G_0 \to k\}$ of set maps from G_0 to k. It has a natural k-algebra structure. The functor

$$G : \underline{k\text{-Alg}} \to \underline{\text{Set}}, \quad G(R) = \hom_k(A, R)$$

has a natural group structure and for R without nontrivial idempotents $G(R) \cong G_0$ This group scheme is called the **constant group scheme**.

Problem 3. Fill in the details of the previous example.

5.2 More ways to build group schemes

We have already seen how given two affine group schemes G, H, we can form the product affine group scheme $G \times H$. We next discuss several other methods for creating new affine group schemes from old ones.

5.2.1 Fibered products

Let $\psi_1 : G_1 \to H$ and $\psi_2 : G_2 \to H$ be morphisms of group schemes. Then we can form the fibered product

$$G_1 \times_H G_2(R) = G_1(R) \times_{H(R)} G_2(R) = \{(g_1, g_2) \in G_1(R) \times G_2(R) | \psi_1(g_1) = \psi_2(g_2)\}.$$

This set has the structure of a group in an obvious way. It is represented as a scheme by $A_1 \otimes_B A_2$, where here A_i represents G_i and B represents H.

5.2.2 Limits of affine group schemes

Given a functor $F : I \to \underline{\text{Aff. Grp}}$, the limit (ie. inverse limit)

$$\varprojlim F : R \mapsto \varprojlim F(R)$$

exists and has the structure of an affine group.

Example 26. Consider the chain of group homomorphisms

$$\cdots \to \mu_{p^3} \to \mu_{p^2} \to \mu_p \to 0.$$

Then there is a unique group scheme G with morphisms $\psi_n : G(R) \to \mu_{p^n}(G)$ satisfying

$$p\psi_{n+1}(g) = \psi_n(pg).$$

It is represented by the ring of formal series

$$A = \left\{ \sum_{j=0}^{\infty} a_j(x)x^{pj} : \deg(a_j(x)) < p. \right\}.$$

5.2.3 Extension and restriction of scalars

If k' is a k-algebra, we can extend a group scheme G over k to a group scheme $G_{k'}$ over k' by defining

$$G_{k'} : R \mapsto G(R),$$

where the above expression makes sense, since any k' algebra is automatically a k-algebra. The functor $G_{k'}$ is represented by the k'-algebra $k' \otimes_k A$, where A represents G. We call $G_{k'}$ an **extension of scalars** of G.

Going the other way, we also have a functor called the **(Weyl) restriction of scalars** which is obtained from a group scheme over k'. Given a group scheme G over k', we define

$$G_{k'/k} : \underline{k\text{-Alg}} \to \underline{\text{Set}}, \quad G_{k'/k}(R) = G(k' \otimes_k R).$$

To show that this is a group scheme in certain situations, we require the following lemma

Proposition 27. Let k' be a k-algebra which is finitely generated and projective as a k-module. Then for any k' algebra A', there exists a k-algebra A with a k-algebra monomorphism $A' \to A$ defining a vector space isomorphism

$$\hom_k(A, R) \cong \hom_{k'}(A', R \otimes_k k') \ \text{ for all } k\text{-algebras } R.$$

Proof. Suppose that $A' = k'[x_1, \ldots, x_m]/I'$ Since k' is finitely generated and projective over k, we know for any prime ideal p of k that k'_p is free of finite rank. In particular, we can write $k'_p = k_p e_1 + \cdots + k_p e_n$ for some orthogonal basis of idempotents e_i. Consider the injective k_p-algebra homomorphism

$$\Phi_p : k'_p[x_1, \ldots, x_m] \to k'_p[y_{11}, \ldots, y_{1n}, \ldots, y_{mn}], \quad x_i \mapsto \sum_{j=1}^{n} y_{ij} e_j.$$

Note that for all $f(x_1, \ldots, x_m)$

$$\Phi_p(f)(y_{11}, \ldots, y_{mn}) = \sum_{j=1}^{n} \Phi_{p,j}(f)(y_{11}, \ldots, y_{mn}) e_j, \quad \Phi_{p,j}(f) \in k_p[y_{11}, \ldots, y_{mn}].$$

Then define the ideal

$$I_p = \{\Phi_{p,j}(f) : 1 \leq j \leq n, \ f \in I'_p\}$$

and consider $A_p = k_p[y_{11}, \dots, y_{mn}]/J_p$. Note that there is a natural k-algebra monomorphism

$$A'_p \to A_p, \quad f(x_1, \dots, x_m) \mapsto \Phi_p(f)(y_{11}, \dots, y_{mn}),$$

which induces an algebra homomorphism

$$\mathrm{Hom}_{k_p}(A_p, R) \to \mathrm{Hom}_{k_p}(A'_p, R) \to \mathrm{Hom}_{k'_p}(A'_p, R \otimes_k k').$$

Noting that

$$\begin{aligned}
\mathrm{Hom}_{k_p}(A_p, R) &= \{(r_{11}, \dots, r_{mn}) \in R : \Phi_p(f)(r_{ij}) = 0 \ \forall f \in I'_p\} \\
&= \{(r_{11}, \dots, r_{mn}) \in R : f(r_i) = 0 \ \forall f \in I'_p, \ r_i = \sum_j r_{ij} e_j\} \\
&= \{(r_1, \dots, r_m) \in R \otimes_k k' : f(r_i) = 0 \ \forall f \in I'_p\} = \mathrm{Hom}_{k'_p}(A'_p, R \otimes_k k').
\end{aligned}$$

we see that the above map is an isomorphism.

Now since (direct) limits exist in the category of k-algebras, we can choose a k-algebra A such that $A \otimes_k k_p \cong A_p$ for all p. The maps Φ_p glue together to a map $\Phi : A' \to A'$ localizing to Φ_p for all p. □

Corollary 28. Suppose that k' is finitely generated and projective as a k-module. Then the functor G is an affine group scheme over k. Furthermore, for any group scheme H over k, we have a bijection

$$\mathrm{Hom}_k(H, G_{k'/k}) \cong \mathrm{Hom}_{k'}(H_{k'}, G)$$

which is natural in G and H.

Proof. This follows immediately from the previous proposition. □