Math 300 Section D
Summer 2014
Final
July 26, 2014
Time Limit: 60 Minutes

Name (Print): ____________________

Student ID: ____________________

This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam. However, you may use a *basic* calculator.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- **Box Your Answer** where appropriate, in order to clearly indicate what you consider the answer to the question to be.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total:	60	

1. (10 points) For each of the following, give an example of a relation satisfying the specified properties. Prove that it satisfies these properties
   (a) (5 points) a relation that is reflexive and symmetric but not transitive
   (b) (5 points) a relation that is symmetric and transitive but not reflexive

Solution 1.

(a) Let R be the relation on $A = \{2, 3, \dots\}$ defined by

$$R = \{(a, b) \in A \times A : a, b \text{ have a common divisor in } A\}$$

Then for all $a \in A$, aRa since $a|a$. Thus R is reflexive. Furthermore, for $a, b \in A$ there's no difference between the common divisors of a and b versus the common divisors of b and a, so it's clear that R is symmetric. However, $2R6$ (since 2 divides both) and $6R3$ (since 3 divides both) but nothing in A divides both 2 and 3, so $\neg(2R3)$. Hence R is not transitive.

(b) Let R be the relation on $\mathbb{R}$ defined by

$$R = (\mathbb{R} \times \mathbb{R}) \backslash \{(0, 0)\}.$$

Then R is not reflexive, since $\neg(0R0)$. However, it is easy to check that R is both symmetric and transitive. The work is left as an exercise for the reader.

2. (10 points) Let A, B be sets. Prove that

$$\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$$

Solution 2. We note the string of equivalences

$$\begin{aligned} X \in \mathcal{P}(A \cap B) &\text{ iff } X \subseteq A \cap B \\ &\text{ iff } (X \subseteq A) \text{ and } (X \subseteq B) \\ &\text{ iff } X \in \mathcal{P}(A) \text{ and } X \in \mathcal{P}(B) \\ &\text{ iff } X \in (\mathcal{P}(A) \cap \mathcal{P}(B)). \end{aligned}$$

Thus $X \in \mathcal{P}(A \cap B)$ if and only if $X \in (\mathcal{P}(A) \cap \mathcal{P}(B))$, and consequently

$$\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B).$$

3. (10 points) Let A and B be sets. State the definition of each of the following phrases. (Note: when stating definitions, precision counts!)
   (a) (2 points) a *relation from* A to B
   (b) (2 points) a *function* from A to B
   (c) (2 points) an *equivalence relation* on A
   (d) (2 points) a *injective function* from A to B
   (e) (2 points) the sets A and B are *equinumerous*

Solution 3.

(a) A relation from A to B is a subset of the cartesian product $A \times B$

(b) A function from A to B is a relation f from A to B satisfying the property that

$$\forall a \in A \exists! b \in B (f(a) = b).$$

(c) An equivalence relation on A is a relation from A to A that is reflexive, symmetric, and transitive

(d) An injective function from A to B is a function $f : A \to B$ satisfying the additional property that

$$\neg(\exists a \in A \exists a' \in A((a \neq a') \wedge (f(a) = f(a')))).$$

(e) Two sets A and B are equinumerous if there exists a function $f : A \to B$ which is bijective.

4. (10 points) Use induction to show that for all integers $n > 0$,

$$8|(5^n + 12n - 1).$$

Proof. (Induction).
(Base case): Let $n = 1$. Then $5^n + 12n - 1 = 16$, which is divisible by 8.
(Inductive step): As an inductive assumption, assume that for some integer $n > 0$,

$$8|(5^n + 12n - 1).$$

Then there exists an integer k such that $5^n + 12n - 1 = 8k$. Thus $5^n = 8k - 12n + 1$ and therefore

$$\begin{aligned} 5^{n+1} + 12(n+1) - 1 &= 5 \cdot 5^n + 12n + 11 \\ &= 5 \cdot (8k - 12n + 1) + 12n + 11 \\ &= 8 \cdot 5k - 5 \cdot 12n + 5 + 12n + 11 \\ &= 8 \cdot 5k - 4 \cdot 12n + 16 \\ &= 8 \cdot 5k - 8 \cdot 6n + 8 \cdot 2 \\ &= 8(5k - 6n + 2). \end{aligned}$$

Thus $5^{n+1} + 12(n+1) - 1$ is divisible by 8. Hence by the principle of mathematical induction, our theorem is true. □

5. (10 points) **True or False Section:**

- For each of the following determine if the statement is true or false.
- If it is true, write TRUE. If it is false write FALSE.
- If you write T or F, or anything else it will be considered a non-response.

(a) (2 points) f^{-1} is a function if and only if f is injective

(b) (2 points) a set A and its power set $\mathcal{P}(A)$ are never equinumerous

(c) (2 points) if A and B are finite sets, then a function $f : A \to B$ is injective if and only if it is surjective

(d) (2 points) if R is a total ordering of a set A, then every subset of A has an R-smallest element

(e) (2 points) if R is an equivalence relation on a set A and $a \in A$, then $x \in [a]_R \Rightarrow [x]_R = [a]_R$

Solution 4.

(a) FALSE

(b) TRUE

(c) TRUE

(d) FALSE

(e) TRUE

6. (10 points) Let A, B, C be sets, and $f : A \to B$ and $g : B \to C$. Consider each of the following statements. If the statement is true, prove it. If the statement is false, provide a counter-example.

   (a) (5 points) If $g \circ f$ is one-to-one, then so too is g

   (b) (5 points) If $g \circ f$ is one-to-one, then so too is f

   Solution 5.

   (a) This is false. For a counter-example, let $A = \{1\}$ and $B = C = \mathbb{R}$. Also let $f : A \to B$ be the function $f(x) = x$, and let $g : B \to C$ be the function $g(x) = x^2$. Then $g(1) = 1 = g(-1)$, so g is not one-to-one. However, $g \circ f : \{1\} \to \mathbb{R}$ is one-to-one, since $\{1\}$ only has a single element to begin with!

   (b) This is true. The proof follows.

   Proof. Suppose that $g \circ f$ is one-to-one. Also let $a, a' \in A$ be arbitrary, and assume that $f(a) = f(a')$. Then

$$(g \circ f)(a) = g(f(a)) = g(f(a')) = (g \circ f)(a').$$

   Since $g \circ f$ is one-to-one, this implies that $a = a'$. Since a, a' were arbitrary elements of A, this shows that f is one-to-one. $\square$