Math 300 Section D
Summer 2014
Exam 1
July 26, 2014
Time Limit: 60 Minutes

Name (Print): ____________________

Student ID: ____________________

This exam contains 8 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam. However, you may use a *basic* calculator.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- **Box Your Answer** where appropriate, in order to clearly indicate what you consider the answer to the question to be.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total:	60	

1. (10 points) In each of the following questions, show that the two statements are equivalent by finding a sequence of equivalent expressions connecting the two. Each equivalence must be justified by an appropriate identity. (No credit for truth table)

   (a) (5 points) $(P \vee Q) \vee \neg(\neg P \vee \neg R)$ and $P \vee Q$

   (b) (5 points) $(P \vee \neg(\neg P \wedge \neg Q)) \wedge \neg((\neg P \wedge R) \vee (R \wedge \neg R))$ and $P \vee (Q \wedge \neg R)$

Solution 1.

(a)

$$\begin{aligned}
(P \vee Q) \vee \neg(\neg P \vee \neg R) &\equiv (P \vee Q) \vee (P \wedge R) && \text{(de Morgan)}\\
&\equiv (Q \vee P) \vee (P \wedge R) && \text{(commutativity)}\\
&\equiv Q \vee (P \vee (P \wedge R)) && \text{(associativity)}\\
&\equiv Q \vee P && \text{(absorbtion)}\\
&\equiv P \vee Q && \text{(commutativity)}
\end{aligned}$$

(b)

$$\begin{aligned}
(P \vee \neg(\neg P \wedge \neg Q)) \wedge \neg((\neg P \wedge R) \vee (R \wedge \neg R)) &\equiv (P \vee \neg(\neg P \wedge \neg Q)) \wedge \neg(\neg P \wedge R) && \text{(contradiction)}\\
&\equiv (P \vee (P \vee Q)) \wedge \neg(\neg P \wedge R) && \text{(de Morgan)}\\
&\equiv (P \vee (P \vee Q)) \wedge (P \vee \neg R) && \text{(de Morgan)}\\
&\equiv (P \vee Q) \wedge (P \vee \neg R) && \text{(idempotent)}\\
&\equiv P \vee (Q \wedge \neg R) && \text{(distributivity)}
\end{aligned}$$

2. (10 points) Let A, B, C be sets. Prove each of the following identites
   (a) (5 points) $A \cap (B \cup (A \cap B)) = A \cap B$
   (b) (5 points) $(A \cap B) \backslash (B \cap C) = (A \cap B) \backslash C$

Solution 2.

Proof of (a). Suppose that $x \in A \cap (B \cup (A \cap B))$. Then by definition $x \in A$ and $x \in B \cup (A \cap B)$. The fact that $x \in B \cup (A \cap B)$ implies that $x \in B$ or $x \in A \cap B$. Note that if $x \in A \cap B$, then in particular $x \in B$. Thus the fact that $x \in B \cup (A \cap B)$ implies that $x \in B$. Hence $x \in A$ and $x \in B$, and therefore $x \in A \cap B$. Since x was an arbitrary element of $A \cap (B \cup (A \cap B))$, this proves that $A \cap (B \cup (A \cap B)) \subseteq A \cap B$.

Next, suppose that $x \in A \cap B$. Then by definition, $x \in A$ and $x \in B$. The fact that $x \in B$ in particular implies that $x \in B \cup (A \cap B)$. Thus since $x \in A$ and $x \in B \cup (A \cap B)$, we see that $x \in A \cap (B \cup (A \cap B))$. Since x was an arbitrary element of $A \cap B$, this proves $A \cap B \subseteq A \cap (B \cup (A \cap B))$. Combining this with the result of the previous paragraph, we conclude that $A \cap (B \cup (A \cap B)) = A \cap B$. ☐

Alternative proof of (a). We note the following chain of logical equivalences

$$
\begin{aligned}
x \in A \cap (B \cup (A \cap B)) &\text{iff } (x \in A) \wedge (x \in B \cup (A \cap B)) \\
&\text{iff } (x \in A) \wedge ((x \in B) \vee (x \in (A \cap B))) \\
&\text{iff } (x \in A) \wedge ((x \in B) \vee ((x \in A) \wedge (x \in B))) \\
&\text{iff } (x \in A) \wedge (((x \in B) \vee (x \in A)) \wedge ((x \in B) \vee (x \in B))) && \text{(distributivity)} \\
&\text{iff } (x \in A) \wedge (((x \in B) \vee (x \in A)) \wedge (x \in B)) && \text{(idempotent)} \\
&\text{iff } (x \in A) \wedge (x \in B) && \text{(absorbtion)} \\
&\text{iff } x \in A \cap B
\end{aligned}
$$

Thus for all elements x, $x \in A \cap (B \cup (A \cap B))$ if and only if $x \in A \cap B$. Hence $A \cap (B \cup (A \cap B)) = A \cap B$. ☐

Proof of (b). Suppose that $x \in (A \cap B) \backslash (B \cap C)$. Then $x \in A \cap B$ and $x \notin (B \cap C)$. Since $x \notin B \cap C$, we know that $x \notin B$ or $x \notin C$. However, since $x \in A \cap B$ we must have that $x \in B$. Therefore $x \notin C$. Thus since $x \in A \cap B$ and $x \notin C$, we conclude that $x \in (A \cap B) \backslash C$. Since x was an arbitrary element of $(A \cap B) \backslash (B \cap C)$, this proves that $(A \cap B) \backslash (B \cap C) \subseteq (A \cap B) \backslash C$.

Next, suppose that $x \in (A \cap B) \backslash C$. Then $x \in A \cap B$ and $x \notin C$. The fact that $x \notin C$ implies that $x \notin B \cap C$. Thus, since $x \in A \cap B$ and $x \notin B \cap C$, we may conclude that $x \in (A \cap B) \backslash (B \cap C)$. Since x was an arbitrary element of $(A \cap B) \backslash C$, this porves that $(A \cap B \backslash C \subseteq (A \cap B) \backslash (B \cap C)$. Combining this with the result of the previous paragraph, we conclude that $(A \cap B) \backslash (B \cap C) = (A \cap B) \backslash C$. ☐

Alternative proof of (b). We note the following chain of logical equivalences

$$
\begin{aligned}
x \in (A \cap B)\backslash(B \cap C) &\text{ iff } (x \in (A \cap B)) \wedge (x \notin (B \cap C)) \\
&\text{ iff } (x \in (A \cap B)) \wedge \neg(x \in (B \cap C)) \\
&\text{ iff } ((x \in A) \wedge (x \in B)) \wedge \neg((x \in B) \wedge (x \in C)) \\
&\text{ iff } ((x \in A) \wedge (x \in B)) \wedge ((x \notin B) \vee (x \notin C)) && \text{(de Morgan)} \\
&\text{ iff } (x \in A) \wedge ((x \in B) \wedge ((x \notin B) \vee (x \notin C))) && \text{(associativity)} \\
&\text{ iff } (x \in A) \wedge (((x \in B) \wedge (x \notin B)) \vee ((x \in B) \wedge (x \notin C))) && \text{(distributivity)} \\
&\text{ iff } (x \in A) \wedge ((x \in B) \wedge (x \notin C)) && \text{(contradiction)} \\
&\text{ iff } ((x \in A) \wedge (x \in B)) \wedge (x \notin C) && \text{(associativity)} \\
&\text{ iff } (x \in (A \cap B)) \wedge (x \notin C) \\
&\text{ iff } x \in (A \cap B)\backslash C
\end{aligned}
$$

Thus for all elements x, $x \in (A \cap B)\backslash(B \cap C)$ if and only if $x \in (A \cap B)\backslash C$. Hence $(A \cap B)\backslash(B \cap C) = (A \cap B)\backslash C$. $\square$

3. (10 points) In each of the following questions, show that the two statements are equivalent by finding a sequence of equivalent expressions connecting the two. Each equivalence must be justified by an appropriate identity. (No credit for truth table)

   (a) (5 points) $P \Leftrightarrow (Q \Rightarrow (P \vee R))$ and $P \vee (Q \wedge \neg R)$

   (b) (5 points) $(P \wedge \neg Q) \Rightarrow (P \Rightarrow Q)$ and $\neg P \vee Q$

Solution 3.

(a)

$$\begin{aligned}
P \Leftrightarrow (Q \Rightarrow (P \vee R)) &\equiv P \Leftrightarrow (\neg Q \vee (P \vee R)) && \text{(conditional)}\\
&\equiv (P \wedge (\neg Q \vee (P \vee R))) \vee (\neg P \wedge \neg(\neg Q \vee (P \vee R))) && \text{(biconditional)}\\
&\equiv (P \wedge ((P \vee R) \vee \neg Q)) \vee (\neg P \wedge \neg(\neg Q \vee (P \vee R))) && \text{(commutative)}\\
&\equiv (P \wedge (P \vee (R \vee \neg Q))) \vee (\neg P \wedge \neg(\neg Q \vee (P \vee R))) && \text{(associative)}\\
&\equiv P \vee (\neg P \wedge \neg(\neg Q \vee (P \vee R))) && \text{(absorbtion)}\\
&\equiv P \vee (\neg P \wedge (Q \wedge \neg(P \vee R))) && \text{(de Morgan)}\\
&\equiv P \vee (\neg P \wedge (Q \wedge (\neg P \wedge \neg R))) && \text{(de Morgan)}\\
&\equiv P \vee (\neg P \wedge ((\neg P \wedge \neg R) \wedge Q)) && \text{(commutative)}\\
&\equiv P \vee (\neg P \wedge (\neg P \wedge (\neg R \wedge Q))) && \text{(associative)}\\
&\equiv P \vee (\neg P \wedge (\neg R \wedge Q)) && \text{(absorbtion)}\\
&\equiv (P \vee \neg P) \wedge (P \vee (\neg R \wedge Q)) && \text{(distributivity)}\\
&\equiv P \vee (\neg R \wedge Q) && \text{(tautology)}\\
&\equiv P \vee (Q \wedge \neg R) && \text{(commutativity)}
\end{aligned}$$

(b)

$$\begin{aligned}
(P \wedge \neg Q) \Rightarrow (P \Rightarrow Q) &\equiv \neg(P \wedge \neg Q) \vee (P \Rightarrow Q) && \text{(conditional)}\\
&\equiv \neg(P \wedge \neg Q) \vee (\neg P \vee Q) && \text{(conditional)}\\
&\equiv (\neg P \vee Q) \vee (\neg P \vee Q) && \text{(de Morgan)}\\
&\equiv \neg P \vee Q && \text{(idempotent)}
\end{aligned}$$

4. (10 points) Let $\mathcal{F}$ and $\mathcal{G}$ be families of sets. Prove that

$$\cup(\mathcal{F} \cap \mathcal{G}) \subseteq (\cup\mathcal{F}) \cap (\cup\mathcal{G}).$$

Solution 4.

Proof. Suppose that $x \in \cup(\mathcal{F}\cap\mathcal{G})$. Then there exists a set $A \in \mathcal{F}\cap\mathcal{G}$ with $x \in A$. Furthermore, the fact that $A \in \mathcal{F} \cap \mathcal{G}$ implies that $A \in \mathcal{F}$ and $A \in \mathcal{G}$. Thus since $x \in A$ and $A \in \mathcal{F}$, we have $x \in \cup\mathcal{F}$. Also since $x \in A$ and $A \in \mathcal{G}$ we have $x \in \cup\mathcal{G}$. We have shown that x is in both $\cup\mathcal{F}$ and $\cup\mathcal{G}$, so as a consequence $x \in (\cup\mathcal{F}) \cap (\cup\mathcal{G})$. Since x was an arbitrary element of $\cup(\mathcal{F} \cap \mathcal{G})$, this proves that

$$\cup(\mathcal{F} \cap \mathcal{G}) \subseteq (\cup\mathcal{F}) \cap (\cup\mathcal{G}).$$

$\square$

5. (10 points) For each of the following statements, either prove the statement or provide a counter-example.

   (a) (5 points) Let n be an integer. Then n is divisible by 12 if and only if n is divisible by 4 and n is divisible by 3.

   (b) (5 points) Let n be an integer. Then n is divisible by 18 if and only if n is divisible by 3 and n is divisible by 10.

Solution 5.

(a)

Proof of (a).
($\Leftarrow$) Suppose that n is divisible by 12. Then there exists an integer k with $n = 12k$. In particular, this shows us that $n = 4(3k)$ and $n = 3(4k)$, so n is divisible by 4 and 3, respectively.

($\Rightarrow$) Suppose that n is divisible by 4 and 3. Then there exist integers j and k with $n = 3j$ and $n = 4k$. It follows that

$$k = 4k - 3k = n - 3k = 3j - 3k = 3(j - k).$$

Hence $k = 3(j - k)$. Consequently, we see that

$$n = 4k = 4(3(j - k)) = 12(j - k).$$

This shows that n is divisible by 12. □

(b) The statement of (b) is false. For a counter-example, take $n = 30$. Then n is divisible by 3 and 10, but not divisible by 18.

6. (10 points) Let n be an integer. Prove that n^4 is divisible by 8 or $n^4 - 1$ is divisible by 8.

Solution 6.

Proof. We consider two cases:

Case 1 (n is even). Assume n is even. Then $n = 2k$ for some integer k. Therefore

$$n^4 = 16k^4 = 8(2k^4),$$

which is divisible by 8.

Case 2 (n is odd). Assume n is odd. Then $n = 2k + 1$ for some integer k. Therefore

$$n^4 = 16k^4 + 32k^3 + 24k^2 + 8k + 1,$$

so that

$$n^4 - 1 = 16k^4 + 32k^3 + 24k^2 + 8k = 8(2k^4 + 4k^3 + 3k^2 + k),$$

which is divisible by 8.

In either case, either n^4 is divisible by 8 or $n^4 - 1$ is divisible by 8. Since n must be even or odd, this completes the proof. □