MATH 300: Problem Set #6

Due on: August 22, 2014

Problem 1 *Velleman Problems*

- pp. 265-267 # 2,3,8,9,11,13,16
- pp. 295-300 # 4,5
- pp. 312-315 # 3,5,11,15,16

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Problem 2 *Cantor's Theorem*

In this problem, we will prove Cantor's Theorem

Theorem 1 (Cantor)**.** Suppose that A is a set. Then A and $\mathcal{P}(A)$ are not equinumerous.

Note that the theorem makes no assumptions about the set A (such as assuming that it is finite). As a consequence, this theorem implies the existence of uncountable sets; in particular, it shows that the power set $\mathcal{P}(\mathbb{Z}^+)$ of $\mathbb{Z}^+$ is uncountable.

Complete the proof of Cantor's theorem below, by filling in the justifications of the two claims in the proof below.

Proof. To show that A and $\mathcal{P}(A)$ are not equinumerous, it suffices to prove that there does not exist an onto function from A to $\mathcal{P}(A)$. To prove this, we assume otherwise and will arrive at a contradiction.

Let $f : A \to \mathcal{P}(A)$ be an arbitrary function from A to $\mathcal{P}(A)$. Assume that f is onto. Let $B \in \mathcal{P}(A)$ be the set

$$B = \{a \in A : a \notin f(a)\}.$$

Then since f is onto, there exists an element $b \in A$ such that $f(b) = B$.

Claim 1: if $b \in B$, then we get a contradiction

Claim 2: if $b \notin B$, then we get a contradiction

Thus since either $b \in B$ or $b \notin B$, we have a contradiction. This shows that our original assumption, that f is onto, must be false. Since f was arbitrary, this completes our proof. □

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Problem 3 *Tacking on an element...*

Suppose that A is a countable set and b is some element that is not a member of A. Prove that if A is not a finite set, then A and $A \cup \{b\}$ are equinumerous.

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