Math 307 Section N
Fall 2014
Exam 1
October 20, 2014
Time Limit: 50 Minutes

Name (Print): ____________________

Student ID: ____________________

This exam contains 6 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam. However, you may use a single, handwritten, one-sided notesheet and a *basic* calculator.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- **Box Your Answer** where appropriate, in order to clearly indicate what you consider the answer to the question to be.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. (10 points) (a) (5 points) Find the general solution of the following first-order linear ODE using variation of parameters

$$ty' = y + 7t^2 e^{2t}$$

   (b) (5 points) Find a solution to the following IVP using the method of integrating factors

$$y' - 2y = \cos(t), \quad y(0) = 2$$

Solution 1.

(a) In order to use the method of variation of parameters, we must first put our equation in the $y' = p(t)y + q(t)$. Therefore, we divide by t:

$$y' = \frac{1}{t}y + 7te^{2t}.$$

In particular, $p(t) = 1/t$ and $q(t) = 7te^{2t}$. Next, we write down the corresponding homogeneous equation

$$y_h' = \frac{1}{t}y_h$$

This equation is separable; a solution is $y_h = t$. Then from the formula for variation of parameters, we know that the solution is

$$y = y_h \int \frac{q}{y_h}dt = t\int \frac{7te^{2t}}{t}dt = t\int 7e^{2t}dt = \frac{7}{2}te^{2t} + Ct.$$

(b) For this equation, an integrating factor is given by

$$\mu(t) = e^{\int -2dt} = e^{-2t}.$$

Multiplying by the integrating factor, we obtain the exact equation

$$y'e^{-2t} - 2ye^{-2t} = e^{-2t}\cos(t),$$

which using the usual first-order exact equation kung foo becomes

$$(ye^{-2t})' = e^{-2t}\cos(t)$$

Then integrating both sides with respect to t, we find

$$ye^{-2t} = \int e^{-2t}\cos(t)dt = \frac{-2}{5}e^{-2t}\cos(t) + \frac{1}{5}e^{-2t}\sin(t) + C.$$

Therefore we see

$$y = \frac{-2}{5}\cos(t) + \frac{1}{5}\sin(t) + Ce^{2t},$$

and since $y(0) = 2$, we find $C = 12/5$, and therefore

$$y = \frac{-2}{5}\cos(t) + \frac{1}{5}\sin(t) + \frac{12}{5}e^{2t}.$$

2. **Multiple Choice Section!**

Directions: The multiple choice section consists of five multiple choice questions. You are NOT required to justify your answer.

(a) (2 points) What kind of differential equation is

$$t\frac{dy}{dt} = 3(y+1)^2 + 2$$

A. Linear B. Separable C. Homogeneous D. Autonomous

(b) (2 points) Four banks offer you four different CDs (a kind of long-term savings account), with differing rates of return. Which of the following four interest rates will give you the largest return on investment in 20 years?
A. 1.80 percent compounded yearly
B. 1.85 percent compounded monthly
C. 1.85 percent compounded continuously
D. 1.80 percent compounded quarterly

(c) (2 points) Consider the initial value problem

$$y' = f(t,y), \quad y(0) = 1.$$

For which of the following choices of function f should we not expect a unique solution?
A. $f(t,y) = t+y$ B. $f(t,y) = e^{ty+y}$ C. $f(t,y) = y^{1/3}$ D. $f(t,y) = (1-y)^{4/5}$

(d) (2 points) Consider the initial value problem

$$\sin(t)y + ty' = \frac{1}{t-4}, \quad y(\pi) = 1$$

What is the largest interval on which we should expect a unique solution to be defined?
A. $(-\infty, 4)$ B. $(0, 4)$ C. $(4, \infty)$ D. $(0, \pi)$

(e) (2 points) Which of the following statements is FALSE for the differential equation

$$y' = -y^3 + 4y.$$

A. The solution is $y = -\frac{1}{4}y^4 + 2y^2 + C$
B. The equation has two stable equilibrium points
C. A solution satisfying $y(0) = 0.1$ will tend to 2 as $t \to \infty$
D. The equation has one unstable equilibrium point

Solution 2. B,C,D,B,A

3. (10 points) (a) (5 points) First show that the equation

$$\cos(x+y) + (\cos(x+y) + e^y)y' = 0$$

is exact. Then solve it.

(b) (5 points) Find an integrating factor for the equation

$$y + (x + xy)y' = 0$$

You need NOT solve it.

Solution 3.

(a) We calculate $\frac{\partial M}{\partial y} = -\sin(x+y)$ and also $\frac{\partial N}{\partial x} = -\sin(x+y)$.

(b) We guess an integrating factor of the form $\mu(x,y) = \mu(y)$. Then the equation

$$\mu(y)y + \mu(y)(x+xy)y' = 0$$

must be exact. This means that

$$\frac{\partial}{\partial y}(\mu(y)y) = \mu'(y)y + \mu(y)$$

and

$$\frac{\partial}{\partial x}(\mu(y)(x+xy)) = \mu(y)(1+y)$$

must be equal. In other words

$$\mu'(y)y + \mu(y) = \mu(y)(1+y),$$

which simplifies to

$$\mu'(y)y = \mu(y)y,$$

and further simplifies to

$$\mu'(y) = \mu(y).$$

This equation is separable; a solution is $\mu(y) = e^y$, which supplies us with an integrating factor.

4. (10 points) A tank with a capacity of 500 gal originally contains 200 gal of water with 100 lbs of salt in solution. Two pipes allow saltwater to flow into the tank and a third pipe allows the resultant solution to flow out again. Water containing a lbs of salt per gallon flows into the tank at a rate of 2 gal/min through the first pipe. Water containing 1 lb of salt per gallon flows into the tank at a rate of 1 gal/min through the second pipe. A well-mixed solution leaves the tank through the third pipe at a rate of 3 gal/min.

   (a) Determine the amount $S(t)$ of salt in the tank (in lbs) as a function of time (your answer will involve a).

   (b) For what value of a will the *concentration* of salt in the tank approach 5/3 lbs per gallon at large times?

Solution 4.

(a) First of all, note that the rate of liquid coming in is equal to the rate of liquid going out. Therefore the volume of liquid in the tank will remain constant $V = 200$ gallons. Next, from the question we see that the amount S of salt in the tank satisfies the differential equation

$$\frac{dS}{dt} = \overbrace{2a+1}^{\text{rate in}} - 3\overbrace{S/200}^{\text{rate out}}.$$

This equation is both linear and separable, so we have several methods to solve it. The general solution is given by

$$S(t) = 200\frac{2a+1}{3} + Ce^{-3t/200}.$$

The initial condition $S(0) = 100$ further implies that $C = (100 - 200(2a+1)/3)$, and therefore

$$S(t) = 200\frac{2a+1}{3} + (100 - 200(2a+1)/3)e^{-t/200}.$$

(b) The concentration is equal to the salt divided by the volume

$$C(t) = S(t)/V = \frac{2a+1}{3} + (0.5 - (2a+1)/3)e^{-t/200}.$$

Taking the limit as $t \to \infty$, we get

$$\lim_{t\to\infty} C(t) = \frac{2a+1}{3}.$$

To have this equal 5/3, we must take $a = 4$.

5. (10 points)

   (a) (5 points) Find a solution to the initial value problem

$$xy' = x^2 \cos(y/x) + y, \quad y(1) = \pi/2.$$

   (b) (5 points) Consider the initial value problem

$$y' = -y^3(y-1)^4(y+1)^5$$

   What are the equilibrium solutions? What is the value of a solution satisfying $y(0) = -0.1$ at very large times?

Solution 5.

(a) First divide both sides by x, to obtain

$$y' = x\cos(y/x) + y/x.$$

The presence of y/x inspires us to do the substitution, $y = xz$ so that $y' = z + xz'$. Then we have

$$z + xz' = x\cos(z) = z$$

Simplifiying, this becomes

$$z' = \cos(z).$$

This equation is separable in z. Separating and integrating, we obtain

$$\ln|\sec(z) + \tan(z)| = x + C.$$

Substituting back in for z, this becomes

$$\ln|\sec(y/x) + \tan(y/x)| = x + C.$$

Unfortunately there is no choice of C which satisfies the initial condition...

(b) The equilibrium points are $y = 0, y = 1, y = -1$. Note also that $y = 0$ is a stable equilibrium, so the limit of a solution satisfying $y(0) = -0.1$ goes to 0 at large times.