Math 307 Section F
Spring 2013
Exam 1
April 24, 2013
Time Limit: 50 Minutes

Name (Print): ____________________

Student ID: ____________________

This exam contains 10 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam. However, you may use a single, handwritten, one-sided notesheet and a *basic* calculator.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- **Box Your Answer** where appropriate, in order to clearly indicate what you consider the answer to the question to be.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. (10 points) (a) (5 points) Find the general solution of the following first-order linear ODE using variation of parameters

$$y' = y + 7e^{2t}$$

(b) (5 points) Find the general solution of the following IVP using the method of integrating factors

$$y' - 2y = e^{2t}, \quad y(0) = 2$$

Solution. [(a)]

1. The equation is linear of the form $y' = p(t)y + q(t)$ for $p(t) = 1$ and $q(t) = 7e^{2t}$. The corresponding homogeneous equation is $y_h' = y_h$. A solution to this equation is $y_h = e^t$. Therefore by variation of parameters:

$$y = y_h \int \frac{q}{y_h} dt = e^t \int 7e^t dt = 7e^{2t} + Ce^t.$$

2. The equation has an integrating factor of the form $\mu = e^{-2t}$. Multiplying by this, we get the exact equation

$$e^{-2t}y' - 2e^{-2t}y = 1$$

the terms on the left hand side may be rewritten as $(e^{-2t}y)'$, so that

$$(e^{-2t}y)' = 1.$$

Integrating both sides, and then solving for y, we find

$$y = te^{2t} + Ce^{2t}.$$

2. **Multiple Choice Section!**

Directions: The multiple choice section consists of five multiple choice questions. For each question in this section, you are NOT required to justify your answer. Additionally, the following grading will be applied:

- a correct answer will receive +2 pts
- a missing answer will receive 0 pts
- a incorrect answer will receive -1 pts

with negative scores on this problem rounded to zero.

(a) (2 points) What kind of equation is the first-order ODE

$$y' = \frac{y+t}{y-t}$$

A. Linear B. Separable C. Homogeneous D. Magical

(b) (2 points) Which of the following is NOT true about the equation

$$y' = 3y - 7$$

A. It is linear B. It is separable C. It is exact D. It has int. factor $\mu(t) = e^{-3t}$

(c) (2 points) Consider the initial value problem

$$y' = f(t, y), \quad y(0) = 1.$$

For which of the following choices of function f should we not expect a unique solution?
A. $f(t,y) = t + y$ B. $f(t,y) = \frac{1}{1+y} + t$ C. $f(t,y) = \sin(ty)$ D. $f(t,y) = (1-y)^{1/5}$

(d) (2 points) Consider the autonomous equation

$$y' = y^2(1-y)^2$$

Which of the following is TRUE?

A. 0 is a stable equilibrium point
B. Both 0 and 1 are semistable equilibrium points
C. 1 is an unstable equilibrium point
D. 0 is a stable equilibrium point

(e) (2 points) Consider the initial value problem

$$\sin(t)y + \cos(t)y' = \frac{1}{t-3}, \quad y(\pi) = 1$$

What is the largest interval on which we should expect a solution to be defined?
A. $(-\infty, 3)$ B. $(3, -\infty)$ C. $(\pi/2, 3)$ D. $(3, 3\pi/2)$

Solution. [(a)]

1. C
2. C
3. D

4. B
5. D

3. (10 points) (a) (5 points) First show that

$$ye^{xy} + \cos(x) + xe^{xy}y' = 0$$

is an exact equation. Then solve it.

(b) (5 points) Solve the equation

$$e^x dx + (e^x \cot(y) + 2y \csc(y))dy = 0$$

by finding an integrating factor.

Solution. [(a)]

1. Our equation is of the form $M(x, y) + N(x, y)y' = 0$, with

$$M(x, y) = ye^{xy} + \cos(x)$$

and

$$N(x, y) = xe^{xy}$$

Notice that

$$N_x = e^{xy} + xye^{xy} = M_y$$

and therefore the equation is exact. To solve it, we find a function ψ such that $\psi_x = M$ and $\psi_y = N$. The latter equation implies

$$N = \int xe^{xy} \partial y = e^{xy} + g(x),$$

and therefore

$$\psi_x = ye^{xy} + g'(x).$$

Then since $\psi_x = M$, we must have

$$ye^{xy} + g'(x) = ye^{xy} + \cos(x).$$

Therefore $g'(x) = \cos(x)$, so that $g(x) = \sin(x)$. Thus we find $\psi(x, y) = e^{xy} + \sin(x)$. Solutions are then given by $\psi(x, y) = C$, and therefore our final answer is

$$e^{xy} + \sin(x) = C.$$

2. We try an integrating factor of the form $\mu(x, y) = \mu(y)$. Then multiplying our original differential equation by $\mu(y)$, we obtain

$$\overbrace{\mu(y)e^x}^{M(x,y)} + \overbrace{(e^x \mu(y) \cot(y) + 2\mu(y)y \csc(y))}^{N(x,y)} y' = 0.$$

For this to be exact, we must have $M_y = N_x$. Notice that

$$M_y = \mu'(y)e^x$$

and

$$N_x = e^x \mu(y) \cot(y),$$

so that

$$\mu'(y)e^x = e^x\mu(y)\cot(y).$$

Dividing both sides by e^x, this becomes

$$\mu'(y) = \mu(y)\cot(y),$$

which is a separable equation for μ in terms of y. Solving this, we find $\mu(y) = \sin(y)$ is an integrating factor.

Multiplying both sides of our original equation by $\sin(y)$, we obtain the exact equation

$$\overbrace{\sin(y)e^x}^{M(x,y)} + \overbrace{(e^x\cos(y) + 2y)}^{N(x,y)}\,y' = 0.$$

Now we solve this equation by finding $\psi(x, y)$ satisfying $\psi_x = M$ and $\psi_y = N$. Using the same process as in the previous part, we find

$$\psi(x, y) = \sin(x)e^x + y^2.$$

Therefore the usual family of solutions to our equation is given by

$$\sin(x)e^x + y^2 = C.$$

4. (10 points) A tank with a capacity of 500 gal originally contains 200 gal of water with 100 lbs of salt in solution. Water containing 1 lb of salt per gallon is entering at a rate of 3 gal/min, and the mixture is allowed to flow out of the tank at a rate of 2 gal/min. Find the amount of salt in the tank at any time prior to the instant when the solution begins to overflow. Find the concentration (in pounds per gallon) of salt in the tank when it is on the point of overflowing. Compare this concentration with the theoretical limiting concentration if the tank had infinite capacity.

Solution. Since the volume of water per unit time flowing into the equation is different from that flowing out, the volume of water in the tank is not constant. Rather, it satisfies

$$\frac{dV}{dt} = \overbrace{3}^{\text{rate in}} - \overbrace{2}^{\text{rate out}} .$$

Therefore $dV/dt = 1$, and since $V(0) = 200$ we have $V = 200 + t$. Since the capacity of the tank is 500 gallons, the tank will overflow in 300 minutes.

Now the differential equation for the quantity of salt S (in pounds) is given by

$$\frac{dS}{dt} = \overbrace{3 \cdot 1}^{\text{rate in}} - \overbrace{2 \cdot \frac{S}{V}}^{\text{rate out}} .$$

Therefore $dS/dt = 3 - 2S/(200 + t)$, with the initial condition $S(0) = 100$. An integrating factor for this equation is $(200 + t)^2$. Multiplying by this, we get the exact equation

$$(200 + t)^2 S' = 3(200 + t)^2 - 2S(200 + t).$$

We move all the S and S' terms over to the left side

$$2S(200 + t) + (200 + t)^2 S' = 3(200 + t)^2.$$

The left hand side is the same as $((200 + t)^2 S)'$, and therefore our equation is

$$((200 + t)^2 S)' = 3(200 + t)^2.$$

Integrating both sides and solving for S, we obtain

$$S = (200 + t) + C(200 + t)^{-2}.$$

The initial condition implies $100 = 200 + C/(200)^2$, so that $C = -4 \cdot 10^6$. Therefore

$$S = (200 + t) + \frac{-4 \cdot 10^6}{(200 + t)^2}.$$

Evaluating this at the overflow time, we find

$$S(300) = 500 - 16 = 484 \text{ lbs.}$$

The concentration $C = S/V$ at the overflow time is the amount of salt divided by the volume:

$$C(300) = \frac{484}{500} = 0.968 \text{ lb/gal.}$$

However, if the tank had infinite capacity, then it would never overflow. The concentration as a function of time would then be given by

$$C(t) = \frac{S(t)}{V(t)} = 1 + \frac{-4 \cdot 10^6}{(200 + t)^3}.$$

Therefore as $t \to \infty$, $C(t)$ approaches 1 lb/gal.

5. (10 points) An object falling from the sky at high speed experiences a drag force proportional to it's velocity. Thus its velocity v satisfies the equation

$$\frac{dv}{dt} = -g + kv^2,$$

where k is a constant depending (mostly) on the geometry of the falling object, and g is the gravitational acceleration. During the Red Bull Stratos mission, Felix Baumgartner leapt from a helium balloon and fell 39,045 meters, reaching a maximum velocity of Mach 1.25, which is 377 meters per second (approx 843 miles/hr or 1358 km/h).

(a) (2 points) Using the fact that Baumgartner's velocity will have approached an equilibrium solution to the above autonomous equation, estimate the value of k

(b) (6 points) Using the value for k obtained in (a), use Euler's method to complete the entries in the table below approximating Baumgartner's velocity during the first minute of his launch.

time (seconds)	velocity (meters/sec)
0	0.0
7.5	-73.575
15.0	-144.347742037
22.5	-207.136564609
30.0	-258.500922784
37.5	-297.484233931
45.0	
52.5	
60.0	

(c) (2 points) Use the table above to get a rough idea of at what time Baumgartner broke the sound barrier. Then indicate which of the following statements are true. (Note that the speed of sound is about 343 meters per second).

A. Baumgartner broke the sound barrier between 37.5 and 45.0 seconds.

B. Baumgartner broke the sound barrier between 45.0 and 52.5 seconds.

C. Baumgartner broke the sound barrier between 52.5 and 60.0 seconds.

D. Baumgartner broke the sound barrier some time after 60.0 seconds.

Solution. [(a)]

1. The equilibrium solutions to the differential equation describing Baumgartner's velocity under drag are given by $v = v_0$, where v_0 is a solution to $-g + kv_0^2 = 0$. Therefore

$v_0 = \pm\sqrt{g/k}$. Since he is falling, Baumgartner's terminal velocity is given by the negative value

$$v_{\text{term}} = -\sqrt{g/k}.$$

However, the problem above says that this is about -377 meters/sec, and therefore

$$377 = \sqrt{g/k}.$$

Since $g = 9.81$ meters/sec^2, we obtain

$$k = 6.9022 \cdot 10^{-5}.$$

2. Do it
3. Use the table to find your answer.