Math 307 Section F
Spring 2013
Exam 2
May 22, 2013
Time Limit: 50 Minutes

Name (Print): ____________________

Student ID: ____________________

This exam contains 11 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam. However, you may use a single, handwritten, one-sided notesheet and a *basic* calculator.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- **Box Your Answer** where appropriate, in order to clearly indicate what you consider the answer to the question to be.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total:	60	

1. (10 points) Solve the following initial value problem

$$y'' + 6y' + 9y = 0, \quad y(0) = 1, \; y'(0) = 0$$

Solution 1. We first find the roots of the corresponding characteristic polynomial:

$$r^2 + 6r + 9 = 0$$

This polynomial factors as $(r+3)^2$, so we get the root -3 repeated twice. The corresponding general solution is

$$y = (At + B)e^{-3t},$$

from which we calculate

$$y' = (-3At + A - 3B)e^{-3t}.$$

Then since $y(0) = 1$ we get

$$1 = (A(0) + B)e^{-3(0)} = B,$$

and since $y'(0) = 0$ we get

$$0 = (-3A(0) + A - 3B)e^{-3(0)} = A - 3B.$$

Therefore $B = 1$ and $A = 3$, and consequently

$$y = (3t + 1)e^{-3t}.$$

2. **Propose a Solution Section!**

Directions: The "Propose a Solution" section consists of five linear nonhomogeneous equations. For each of these equations, write down the type of function y (with undetermined coefficients) you would try, in order to get a particular solution. *You do NOT need to solve the equations* For example, if the equation were

$$y'' + 2y' + y = e^t,$$

a *correct answer* would be

$$y = Ae^t,$$

and *incorrect answers* would include

$$y = (At + B)e^t, \quad y = At^2e^{2t}, \quad y = Ae^{3t}, \quad y = A\pi^t$$

Each part is worth 2pts:

(a) (2 points)

$$y'' + 2y' + 2y = t^3e^{4t}$$

(b) (2 points)

$$3y'' + 6y' + 3y = (t + 1)e^{-t}$$

(c) (2 points)

$$y'' + 4y' - 5y = e^t$$

(d) (2 points)

$$y'' + 2y' = t^2 + 1$$

(e) (2 points)

$$y'' - 3y' + 2y = e^{2t}$$

Solution 2.

(a) $y_P = (A + Bt + Ct^2 + Dt^3)e^{4t}$

(b) $y_P = (At^3 + Bt^2)e^{-t}$

(c) $y_P = Ate^t$

(d) $y_P = At^3 + Bt^2 + Ct$

(e) $y_P = Ate^{2t}$

3. (10 points)

(a) (5 points) Show that the equation

$$y\cos(xy) + \cos(x) + (x\cos(xy) + 3y^2)y' = 0.$$

is exact. Then solve it.

(b) (5 points) Find an integrating factor for the equation

$$2xe^{x^2} + 3y^2(e^{x^2} + 1)y' = 0$$

You do not need to solve it.

Solution 3.

(a) In this case

$$M(x, y) = y\cos(xy) + \cos(x)$$

and

$$N(x, y) = x\cos(xy) + 3y^2$$

and so we calculate

$$\frac{\partial M}{\partial y} = \cos(xy) - xy\sin(xy)$$

and also

$$\frac{\partial N}{\partial x} = \cos(xy) - xy\sin(xy)$$

This shows that the equation is exact. Therefore there exists a function $\psi(x, y)$ with $\partial\psi/\partial x = M$ and $\partial\psi/\partial y = N$. Therefore

$$\psi(x, y) = \int \frac{\partial\psi}{\partial x}\partial x = \int M(x, y)\partial x = \int y\cos(xy) + \cos(x)\partial x = \sin(xy) + \sin(x) + h(y)$$

for some arbitrary function $h(y)$. Then

$$\frac{\partial\psi}{\partial y} = \frac{\partial}{\partial y}\left(\sin(xy) + \sin(x) + h(y)\right) = x\cos(xy) + h'(y),$$

and also

$$\frac{\partial\psi}{\partial y} = N(x, y) = x\cos(xy) + 3y^2.$$

Therefore

$$x\cos(xy) + h'(y) = x\cos(xy) + 3y^2,$$

and it follows that $h(y) = y^3$ plus a constant (which we may take to be zero). Thus

$$\psi(x,y) = \sin(xy) + \sin(x) + y^3$$

, and we obtain a family of solutions by setting $\psi = C$, ie

$$\sin(xy) + \sin(x) + y^3 = C.$$

(b) We propose an integrating factor of the form $\mu(x,y) = \mu(y)$. Then the equation

$$\overbrace{2x\mu(y)e^{x^2}}^{M(x,y)} + \overbrace{3\mu(y)y^2(e^{x^2}+1)}^{N(x,y)}\,y' = 0$$

is exact, meaning that

$$\frac{\partial M}{\partial y} = 2xe^{x^2}\mu'(y)$$

and also that

$$\frac{\partial N}{\partial x} = 6x\mu(y)y^2e^{x^2}.$$

Therefore

$$2xe^{x^2}\mu'(y) = 6x\mu(y)y^2e^{x^2},$$

and consequently

$$\mu'(y) = 3\mu(y)y^2.$$

This is a separable equation; one solution is

$$\mu(y) = e^{y^3}.$$

4. (10 points)

(a) (4 points) Find a particular solution to the equation

$$y'' + 2y' + y = \cos(t)e^t$$

(b) (2 points) Find a particular solution to the equation

$$y'' + 2y' + y = \sin(t)e^t$$

(c) (2 points) Find a particular solution to the equation

$$y'' + 2y' + y = 4\cos(t)e^t - 7\sin(t)e^t$$

(d) (2 points) Write down the general solution to the equation

$$y'' + 2y' + y = 4\cos(t)e^t - 7\sin(t)e^t$$

Solution 4.

(a) We squigglify the equation, obtaining

$$\widetilde{y}'' + 2\widetilde{y}' + \widetilde{y} = e^{(1+i)t}$$

From this, we propose the particular solution

$$\widetilde{y}_p = Ae^{(1+i)t}$$

which has

$$\widetilde{y}_p' = A(1+i)e^{(1+i)t}$$

and

$$\widetilde{y}_p'' = A(1+i)^2e^{(1+i)t} = A(2i)e^{(1+i)t}.$$

Plugging this into the squigglified equation, we find

$$2iAe^{(1+i)t} + 2(1+i)Ae^{(1+i)t} + Ae^{(1+i)t} = e^{(1+i)t},$$

and therefore

$$(3+4i)Ae^{(1+i)t} = e^{(1+i)t}.$$

It follows that

$$A = \frac{1}{3+4i} = \frac{3-4i}{25} = \frac{3}{25} - \frac{4}{25}i,$$

and therefore

$$\begin{aligned}
\widetilde{y}_p &= \left(\frac{3}{25} - \frac{4}{25}i\right) e^{(1+i)t} \\
&= \left(\frac{3}{25} - \frac{4}{25}i\right) \left(e^t\cos(t) + ie^t\sin(t)\right) \\
&= \frac{3}{25}e^t\cos(t) + \frac{4}{25}e^t\sin(t) + i\left(-\frac{4}{25}e^t\cos(t) + \frac{3}{25}e^t\sin(t)\right).
\end{aligned}$$

Therefore a particular solution is given by

$$y_p = \mathrm{Re}(\widetilde{y}_p) = \frac{3}{25}e^t\cos(t) + \frac{4}{25}e^t\sin(t)$$

(b) We did all the work that we needed to do in part (a). The only difference is that to get the particular solution, we need only take the imaginary part of the $\widetilde{y}_p$ that we found. Doing so, we obtain

$$y_p = \mathrm{Im}(\widetilde{y}_p) = -\frac{4}{25}e^t\cos(t) + \frac{3}{25}e^t\sin(t).$$

(c) Let y_a be the particular solution we found in (a) and y_b be the particular solution that we found in (b). To get this particular solution, we just need to take an appropriate linear combination of y_a and y_b. Doing so, we obtain

$$y_p = 4y_a - 7y_b = \frac{8}{5}e^t\cos(t) - \frac{1}{5}e^t\sin(t).$$

(d) To do this part, we need only tack on the general solution to the corresponding homogeneous equation

$$y_h'' + 2y_h' + y_h = 0$$

The characteristic polynomial of this equation is $r^2 + 2r + 1 = (r + 1)^2$, and therefore

$$y_h = (At + B)e^{-t}.$$

The general solution is therefore

$$y_p = \frac{8}{5}e^t \cos(t) - \frac{1}{5}e^t \sin(t) + (At + B)e^{-t}.$$

5. (10 points) Given that $y_1 = e^t$ is a solution to the differential equation

$$ty'' - (t+1)y' + y = 0,$$

use the method of reduction of order to find the general solution of the equation.

Solution 5. We propose a solution of the form $y = vy_1 = ve^t$. Then

$$y' = v'e^t + ve^t$$

and

$$y'' = v''e^t + 2v'e^t + ve^t,$$

and therefore

$$\begin{aligned} 0 &= ty'' - (t+1)y' + y \\ &= t(v''e^t + 2v'e^t + ve^t) - (t+1)(v'e^t + ve^t) + ve^t \\ &= te^t v'' + (1-t)e^t v'. \end{aligned}$$

Therefore substituting $w = v'$, we get

$$(t-1)e^t w = te^t w',$$

which is a separable equation. Solving it, we obtain

$$w = Ate^{-t}.$$

Then since $w = v'$, we obtian

$$v = \int Ate^{-t} = A(-te^{-t} - e^{-t}) + B$$

Hence

$$y = ve^t = -A(t+1) + Be^t,$$

which is the general solution.

6. (10 points) Suppose a certain spring is known to respond to a force of 2 lbs by stretching 6 inches. We attach a mass m to this spring and submerge it in a liquid. Experimentally, it is known that the liquid exerts a damping force on the mass spring system, with damping constant $\gamma = 2$ lbs·s/ft.

   (a) For what values of m is the mass spring system overdamped?

   (b) Suppose that the mass we attach weighs 8 lbs, and that the system is initially contracted 6 inches from its equilibrium and then released. Find the position u of the mass (relative to it's equilibrium position) as a function of time.

Solution 6. Since the spring responds to a 2 lb force by stretching 1/2 a foot, we calculate the spring constant by $2 = (1/2)k$, so that $k = 4$ lbs/ft. Therefore the differential equation for this situation is described by

$$mu'' + 2u' + 4u = 0$$

(a) The above situation becomes overdamped when $\gamma^2 = 4km$. Since $\gamma = 2$ and $k = 4$, this occurs when $m = 1/4$ lb·s^2/ft.

(b) For this situation the mass is 8lbs/32ft/s^2 = 1/4. Therefore the initial value problem for this situation is

$$\frac{1}{4}u'' + 2u' + 4u = 0, \quad u(0) = -1/2, \ u'(0) = 0.$$

The general solution of the corresponding homogeneous equation is

$$u = (At + B)e^{-4t},$$

which has derivative

$$u' = (-4At + A - 4B)e^{-4t}$$

. The initial conditions then tell us

$$B = -1/2, \quad A - 4B = 0$$

Therefore

$$u = (-2t - 1/2)e^{-4t}.$$