Math 307 Section F
Spring 2013
Exam 2
May 22, 2013
Time Limit: 50 Minutes

Name (Print): ______________________

Student ID: ______________________

This exam contains 8 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam. However, you may use a single, handwritten, one-sided notesheet and a *basic* calculator.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- **Box Your Answer** where appropriate, in order to clearly indicate what you consider the answer to the question to be.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total:	60	

1. (10 points) Solve the following initial value problem

$$y'' + 5y' + 3y = 0, \quad y(0) = 1, \; y'(0) = 0$$

Solution. The roots of the characteristic polynomial are $(-5 \pm \sqrt{13})/2)$ so the general solution is

$$y = C_1 e^{(-5+\sqrt{13})t/2} + C_2 e^{(-5-\sqrt{13})t/2}$$

Our first initial condition then says that

$$1 = C_1 + C_2.$$

and since

$$y' = C_1 \frac{1}{2}(-5+\sqrt{13}) e^{(-5+\sqrt{13})t/2} + C_2 \frac{1}{2}(-5-\sqrt{13}) e^{(-5-\sqrt{13})t/2}$$

our second initial condition says

$$0 = C_1 \frac{1}{2}(-5+\sqrt{13}) + C_2 \frac{1}{2}(-5-\sqrt{13}).$$

Using the equation $1 = C_1 + C_2$ we may write $C_1 = 1 - C_2$, so that

$$0 = (1 - C_2)\frac{1}{2}(-5+\sqrt{13}) + C_2 \frac{1}{2}(-5-\sqrt{13}) = \frac{1}{2}(-5+\sqrt{13}) - C_2\sqrt{13}$$

consequently

$$C_2 = \frac{1}{2} - \frac{5}{2\sqrt{13}}$$

and therefore since $C_1 = 1 - C_2$

$$C_1 = \frac{1}{2} + \frac{5}{2\sqrt{13}}.$$

The solution to the initial value problem is therefore

$$y = \left(\frac{1}{2} + \frac{5}{2\sqrt{13}}\right) e^{(-5+\sqrt{13})t/2} + \left(\frac{1}{2} - \frac{5}{2\sqrt{13}}\right) e^{(-5-\sqrt{13})t/2}$$

2. **Propose a Solution Section!**

Directions: The "Propose a Solution" section consists of five linear nonhomogeneous equations. For each of these equations, write down the type of function y (with undetermined coefficients) you would try, in order to get a particular solution. *You do NOT need to solve the equations* For example, if the equation were

$$y'' + 2y' + y = e^t,$$

a *correct answer* would be

$$y = Ae^t,$$

and *incorrect answers* would include

$$y = (At + B)e^t, \quad y = At^2e^{2t}, \quad y = Ae^{3t}, \quad y = A\pi^t$$

Each part is worth 2pts:

(a) (2 points)

$$y'' + 3y' + 2y = e^{4t}$$

(b) (2 points)

$$y'' + 3y' + 2y = (t + 1)e^{-t}$$

(c) (2 points)

$$y'' - 2y' + y = e^t$$

(d) (2 points)

$$y'' + 2y' + 4y = t^2$$

(e) (2 points)

$$y'' - 2y' = e^{2t}$$

Solution.

(a) $y_p = Ae^{4t}$
(b) $y_p = (At^2 + Bt)e^{-t}$
(c) $y_p = At^2e^t$
(d) $y_p = At^2 + Bt + C$
(e) $y_p = Ate^{2t}$

3. (10 points) Solve the equation

$$e^x dx + e^x \cot(y) dy = 0$$

by finding an integrating factor of the form $\mu = \mu(y)$. [Hint: $\int \cot(y)dy = \ln(\sin(y)) + C$]

Solution. Multiplying by $\mu(y)$, we obtain the equation

$$\overbrace{\mu(y)e^x}^{M(x,y)} dx + \overbrace{\mu(y)e^x \cot(y)}^{N(x,y)} dy = 0$$

Then since $\mu(y)$ is supposed to be an integrating factor, this new equation must be exact. Therefore $M_y = N_x$, meaning that

$$\mu'(y)e^x = \mu(y)e^x \cot(y)$$

Dividing both sides by e^x we get the separable equation

$$\mu'(y) = \mu(y)\cot(y).$$

We separate and integrate this equation to solve it, using the hint to integrate $\cot(y)$. A solution is found to be $\mu(y) = \sin(y)$, so we have the exact equation

$$\overbrace{\sin(y)e^x}^{M(x,y)} dx + \overbrace{\cos(y)e^x}^{N(x,y)} dy = 0.$$

Now to find the solution to this exact equation, we wish to find a function $\psi(x, y)$ such that $\psi_x = M$ and $\psi_y = N$. Using the expression $\psi_x = M$, we find via "partial integrals":

$$\psi(x, y) = \int M(x, y)\partial x = \int \sin(y)e^x \partial x = \sin(y)e^x + h(y).$$

Then since $\psi_y = N$, we obtain

$$\cos(y)e^x + h'(y) = \cos(y)e^x.$$

It follows that $h'(y) = 0$, and therefore h is constant, which we may take to be 0. Therefore $\psi(x, y) = \sin(y)e^x$, and the solution to the original equation is obtained by setting $\psi(x, y) = C$, ie.

$$\sin(y)e^x = C$$

4. (10 points)

   (a) (4 points) Find a particular solution to the equation

$$y'' + 2y' + y = \cos(t)$$

   (b) (2 points) Find a particular solution to the equation

$$y'' + 2y' + y = \sin(t)$$

   (c) (2 points) Find a particular solution to the equation

$$y'' + 2y' + y = 2\cos(t) - 3\sin(t)$$

   (d) (2 points) Write down the general solution to the equation

$$y'' + 2y' + y = 2\cos(t) - 3\sin(t)$$

Solution.

(a) First, we write down the associated complex ("squigglified") equation

$$\widetilde{y}'' + 2\widetilde{y}' + \widetilde{y} = e^{it}.$$

To solve this equation, we propose a solution of the form $\widetilde{y}_p = Ae^{it}$. Then we calculate $\widetilde{y}_p' = iAe^{it}$ and $\widetilde{y}_p'' = -Ae^{it}$ so that

$$\widetilde{y}_p'' + 2\widetilde{y}_p' + \widetilde{y}_p = 2iAe^{it}.$$

Therefor for $\widetilde{y}_p$ to be a solution to the complex equation, we should take $A = 1/(2i) = -i/2$. Thus we have

$$\widetilde{y}_p = -\frac{1}{2}ie^{it} = \frac{1}{2}\sin(t) - i\frac{1}{2}\cos(t).$$

A particular solution to the original equation in (a) is then $y_p = \mathrm{Re}(\widetilde{y}_p) = \frac{1}{2}\sin(t)$.

(b) Using the work from part (a), we know right away that a particular solution will be

$$y_p = \mathrm{Im}(\widetilde{y}_p) = -\frac{1}{2}\cos(t)$$

(c) Let y_1 be the particular solution found in (a) and y_2 be the particular solution found in (b). Then a particular solution to the equation of (c) will be

$$y_p = 2y_1 - 3y_2 = \sin(t) + \frac{3}{2}\cos(t).$$

(d) To get the general solution, first notice that the general solution to the corresponding homogeneous equation

$$y_h'' + 2y_h' + y_h = 0$$

is

$$y_h = (At + B)e^{-t}.$$

Therefore the general solution to our equation is obtained by adding the particular solution found in (c).

$$y = y_h + y_p = (At + B)e^{-t} + \sin(t) + \frac{3}{2}\cos(t).$$

5. (10 points) Given that $y_1 = t^2$ is a solution to the differential equation

$$t^2y'' - 4ty' + 6y = 0,$$

use the method of reduction of order to find a second solution to the equation.

Solution. We propose a solution of the form $y(t) = v(t)y_1(t)$. Then

$$y' = v'y_1 + vy_1' = v't^2 + 2tv$$

and

$$y'' = v''y_1 + 2v'y_1' + vy_1'' = t^2v'' + 4tv' + 2v$$

so that

$$t^2y'' - 4ty' + 6y = t^4v''.$$

Since $t^2y'' = 4ty' + 6y = 0$, this means that $t^4v'' = 0$, and therefore $v'' = 0$, making $v = At + B$. Thus the general solution of the original equation is given by

$$y = vy_1 = At^3 + Bt^2,$$

which in particular gives us a second solution to the equation.

6. (10 points) A mass weighing 3 lbs stretches a spring 3 inches. If the mass is pushed upward, *contracting* the spring a distance of 1 in., and then set in motion with a downward velocity of 2 ft/s, and if there is no damping, find the position u of the mass at any time. [Caution: Watch your units!]

Solution. A weight of 3 lbs stretches the spring 3 inches, ie. 1/4 of a foot, we calculate the spring constant k to be 12 lbs/ft. Also, there is assumed to be no damping, so $\gamma = 0$. Lastly, the mass on the spring is the weight of 3 lbs divided by the acceleration due to gravity (32 ft/s^2). Therefore the mass is $m = 3/32$ (in units of lb·s^2/ft).

Remember, with a spring system the spring is stretched downward and contracted upward. Therefore the initial condition is $u(0) = -1/12$ and $u'(0) = 2$. The initial value problem we are trying to solve is therefore

$$\frac{3}{32}u'' + 12u = 0, \quad u(0) = -\frac{1}{12}, \quad u'(0) = 2.$$

To solve this problem we first write down the general solution to the original equation

$$u = A\cos\left(\sqrt{128}t\right) + B\sin\left(\sqrt{128}t\right)$$

Our first initial condition then says

$$-\frac{1}{12} = A$$

and since

$$u' = -\sqrt{128}A\sin\left(\sqrt{128}t\right) + \sqrt{128}\cos\left(\sqrt{128}t\right)$$

our second equation says

$$2 = \sqrt{128}B$$

so that $B = \frac{2}{\sqrt{128}}$ and therefore

$$u = -\frac{1}{12}\cos\left(\sqrt{128}t\right) + \frac{2}{\sqrt{128}}\sin\left(\sqrt{128}t\right)$$