

MATH 307: Problem Set #1

Due on: April 5, 2013

Problem 1 *First Order Linear Equations*

For each of the following equations:

- (i) Use a computer to graph the slope field of the differential equation.
Include a printout of your graph with your homework
 - (ii) Based on inspection of the direction field, describe how you expect solutions to behave for large values of t
 - (iii) Find the general solution to the equation and use it to determine how the solution behaves as $t \rightarrow \infty$.
- (a) $y' - 2y = t^2 e^{2t}$
 - (b) $ty' + 2y = \sin(t)$, for $t > 0$
 - (c) $ty' - y = t^2 e^{-t}$, for $t > 0$

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Solution 1.

- (a) Suppose that $\mu = \mu(t)$ is our integrating factor. Then

$$\mu(t)y' - 2\mu(t)y = \mu(t)t^2 e^{2t}$$

is exact. Putting this in M, N -notation

$$\overbrace{-\mu(t)t^2 e^{2t} - 2\mu(t)y}^{M(t,y)} + \overbrace{\mu(t)}^{N(t,y)} y' = 0$$

exactness means that $\partial M/\partial y = \partial N/\partial t$, and therefore

$$-2\mu(t) = \mu'(t)$$

This differential equation is easy to solve! A solution is $\mu(t) = e^{-2t}$. Thus the equation

$$e^{-2t}y' - 2e^{-2t}y = t^2$$

is exact! We can group stuff together on the left hand side to then get

$$(e^{-2t}y)' = t^2.$$

Now integrating both sides with respect to t , we find

$$\begin{aligned}\int (e^{-2t}y)' dt &= \int t^2 dt \\ e^{-2t}y &= \frac{1}{3}t^3 + C \\ y &= \frac{1}{3}t^3 e^{2t} + C e^{2t}\end{aligned}$$

(b) Suppose that $\mu = \mu(t)$ is our integrating factor. Then

$$\mu(t)ty' + 2y\mu(t) = \mu(t)\sin(t)$$

is exact. Putting this into M, N notation, we then get

$$\overbrace{2\mu(t)y - 2\mu(t)\sin(t)}^{M(t,y)} + \overbrace{\mu(t)t y'}^{N(t,y)} = 0.$$

exactness means that $\partial M/\partial y = \partial N/\partial t$, and therefore

$$2\mu(t) = \mu(t) + t\mu'(t).$$

This simplifies to

$$\mu(t) = t\mu'(t).$$

This equation really easy! A solution is $\mu(t) = t$. Thus the equation

$$t^2y' + 2ty = t\sin(t)$$

is exact! We can group the things on the left together to find

$$(t^2y)' = t\sin(t)$$

Now integrating both sides with respect to t , we find

$$\begin{aligned}\int (t^2y)' dt &= \int t\sin(t) dt \\ t^2y &= \sin(t) - t\cos(t) + C \\ y &= t^{-2}\sin(t) - t^{-1}\cos(t) + Ct^{-2}\end{aligned}$$

(c) Suppose that $\mu = \mu(t)$ is our integrating factor. Then

$$\mu(t)ty' - \mu(t)y = \mu(t)t^2e^{-t}$$

is exact. Putting this in M, N -notation

$$\overbrace{-\mu(t)y - \mu(t)t^2e^{-t}}^{M(t,y)} + \overbrace{t\mu(t)y'}^{N(t,y)} = 0$$

exactness means that $\partial M/\partial y = \partial N/\partial t$, and therefore

$$-\mu(t) = \mu(t) + t\mu'(t)$$

which simplifies to

$$-2\mu(t) = t\mu'(t).$$

This differential equation is a quickie, and a solution is $\mu(t) = t^{-2}$. Thus the equation

$$t^{-1}y' - t^{-2}y = e^{-t}$$

is exact! We can group things together on the left hand side to get

$$(t^{-1}y)' = e^{-t}$$

Integrating both sides, we find

$$\begin{aligned} \int (t^{-1}y)' dt &= \int e^{-t} dt \\ t^{-1}y &= -e^{-t} + C \\ y &= -te^{-t} + Ct \end{aligned}$$

Problem 2 *First Order Linear Initial Value Problems*

Find the solution to each of the given initial value problems

- (a) $y' + 2y = te^{-2t}, y(1) = 0$
- (b) $y' + 2y/t = \frac{1}{t^2} \cos(t), y(\pi) = 0$
- (c) $ty' + 2y = \sin(t), y(\pi/2) = 1$
- (d) $t^3y' + 4t^2y = e^{-t}, y(-1) = 0$

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Solution 2.

(a) Multiply by integrating factor $\mu(t)$:

$$\mu(t)y' + 2\mu(t)y = \mu(t)te^{-2t}$$

Put into M, N -notation form:

$$\overbrace{2\mu(t)y - \mu(t)te^{-2t}}^{M(t,y)} + \overbrace{\mu(t)}^{N(t,y)} y' = 0$$

Exactness implies that $\partial M/\partial y = \partial N/\partial t$ and therefore

$$2\mu(t) = \mu'(t)$$

A solution of this equation is $\mu(t) = e^{2t}$. Thus we have

$$e^{2t}y' + 2e^{2t}y = t.$$

Gathering together the y -terms, we get

$$(e^{2t}y)' = t.$$

Lastly we integrate and solve for y :

$$\begin{aligned} \int (e^{2t}y)' dt &= \int t dt \\ e^{2t}y &= \frac{1}{2}t^2 + C \\ y &= \frac{1}{2}t^2e^{-2t} + Ce^{-2t} \end{aligned}$$

Since $y(1) = 0$, we get

$$0 = \frac{1}{2}1^2e^{-2(1)} + Ce^{-2(1)},$$

and therefore $C = -\frac{1}{2}$. Thus the solution is

$$y(t) = \frac{1}{2}t^2e^{-2t} - \frac{1}{2}e^{-2t}$$

(b) Multiply by integrating factor $\mu(t)$:

$$\mu(t)y' + 2\mu(t)y/t = \mu(t)\frac{1}{t^2}\cos(t)$$

In M, N notation, this is

$$\overbrace{2\mu(t)y/t - \mu(t)\frac{1}{t^2}\cos(t)}^{M(t,y)} + \overbrace{\mu(t)}^{N(t,y)} y' = 0$$

Exactness implies that $\partial M/\partial y = \partial N/\partial t$ and therefore

$$2\mu(t)/t = \mu'(t)$$

and a solution to this is $\mu(t) = t^2$. Thus we have the exact equation

$$t^2 y' + 2ty = \cos(t)$$

Exactness of this equation allows us to gather all the y parts to get

$$(t^2 y)' = \cos(t)$$

Integrating then leads to

$$\begin{aligned} \int (t^2 y)' dt &= \int \cos(t) dt \\ t^2 y &= \sin(t) + C \\ y &= t^{-2} \sin(t) + C t^{-2} \end{aligned}$$

Since $y(\pi) = 0$, we also know that

$$0 = \pi^{-2} \sin(\pi) + C \pi^{-2}$$

and therefore $C = 0$. The solution is therefore

$$y = t^{-2} \sin(t).$$

(c) Multiply by integrating factor $\mu(t)$:

$$\mu(t) t y' + 2\mu(t) y = \mu(t) \sin(t)$$

In M, N notation form

$$\overbrace{2\mu(t)y - \mu(t) \sin(t)}^{M(t,y)} + \overbrace{\mu(t)t y'}^{N(t,y)} = 0$$

Exactness implies that $\partial M/\partial y = \partial N/\partial t$ and therefore

$$2\mu(t) = \mu(t) + t\mu'(t).$$

This simplifies to

$$\mu(t) = t\mu'(t).$$

A solution of this equation is $\mu(t) = t$. Thus we have the exact equation

$$t^2 y' + 2ty = t \sin(t)$$

Exactness gives us the power to gather together the y parts:

$$(t^2 y)' = t \sin(t).$$

Integrating both sides, we find

$$\begin{aligned}\int (t^2 y)' dt &= \int t \sin(t) dt \\ t^2 y &= \sin(t) - t \cos(t) + C \\ y &= t^{-2} \sin(t) - t^{-1} \cos(t) + Ct^{-2}\end{aligned}$$

Now using the fact that $y(\pi/2) = 1$ we find

$$1 = (\pi/2)^{-2} \sin(\pi/2) - (\pi/2)^{-1} \cos(\pi/2) + C(\pi/2)^{-2},$$

and therefore $C = \frac{\pi^2}{4} - 1$. Thus the solution is

$$y = t^{-2} \sin(t) - t^{-1} \cos(t) + \left(\frac{\pi^2}{4} - 1\right) t^{-2}$$

(d) Multiply by integrating factor $\mu(t)$:

$$\mu(t)t^3 y' + 4\mu(t)t^2 y = \mu(t)e^{-t}$$

In M, N -notation form this is

$$\overbrace{4\mu(t)t^2 y - \mu(t)e^{-t}}^{M(t,y)} + \overbrace{\mu(t)t^3 y'}^{N(t,y)} = 0$$

Exactness implies that $\partial M/\partial y = \partial N/\partial t$ and therefore

$$4\mu(t)t^2 = \mu'(t)t^3 + 3t^2\mu(t)$$

which simplifies to

$$\mu(t) = \mu'(t)t.$$

A solution to this equation is $\mu(t) = t$, and therefore we have the exact equation

$$t^4 y' + 4t^3 y = te^{-t}$$

Since this is exact, we can gather the y parts:

$$(t^4 y)' = te^{-t}$$

Now integrating both sides with respect to t , we find

$$\begin{aligned}\int (t^4 y)' dt &= \int te^{-t} dt \\ t^4 y &= -te^{-t} - e^{-t} + C \\ y &= -t^{-3}e^{-t} - t^{-4}e^{-t} + Ct^{-4}\end{aligned}$$

Using the fact that $y(-1) = 0$, we then obtain

$$0 = -(-1)^{-3}e^1 - (-1)^{-4}e^1 + C(-1)^{-4}$$

and therefore $C = 0$. Hence the solution is

$$y = -t^{-3}e^{-t} - t^{-4}e^{-t}.$$

Problem 3 *Variation of Parameters*

Use variation of parameters to find the general solution of the given differential equation

(a) $y' + y/t = 3 \cos(2t)$

(b) $2y' + y = 3t^2$

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Solution 3.

(a) We rewrite this in $y' = p(t)y + q(t)$ notation:

$$y' = \overbrace{-\frac{1}{t}}^{p(t)} y + \overbrace{3 \cos(2t)}^{q(t)}.$$

Let y_h be a solution to the corresponding homogeneous equation. That is:

$$y'_h = -\frac{1}{t} y_h$$

A solution to this equation is $y_h(t) = \frac{1}{t}$. Then the method of variation of parameters tells us that $y = v(t)y_h(t)$, where

$$v(t) = \int \frac{q(t)}{y_h(t)} dt = \int 3t \cos(2t) dt = \frac{3}{2} t \sin(2t) + \frac{3}{4} \cos(2t) + C$$

Hence

$$y(t) = \frac{3}{2} \sin(2t) + \frac{3}{4} t^{-1} \cos(2t) + C t^{-1}$$

(b) We rewrite this in $y' = p(t)y + q(t)$ notation:

$$y' = \overbrace{-\frac{1}{2}}^{p(t)} y + \overbrace{\frac{3}{2} t^2}^{q(t)}.$$

Let y_h be a solution to the corresponding homogeneous equation. That is:

$$y'_h = -\frac{1}{2} y_h$$

A solution to this equation is $y_h(t) = e^{-t/2}$. Then the method of variation of parameters tells us that $y = v(t)y_h(t)$, where

$$v(t) = \int \frac{q(t)}{y_h(t)} dt = \int \frac{3}{2} t^2 e^{t/2} dt = 3t^2 e^{t/2} - 12t e^{t/2} + 24e^{t/2} + C$$

Hence

$$y(t) = 3t^2 - 12t + 24 + C e^{-t/2}$$

Problem 4 *Separable Equations*

In each of the following, find a family of solutions parametrized by a constant

(a) $y' = \frac{x^2}{y(1+x^3)}$

(b) $y' + y^2 \sin(x) = 0$

(c) $\frac{dy}{dx} = \frac{x-e^{-x}}{y+e^y}$

(d) $\frac{dy}{dx} = \frac{x^2}{1+y^2}$

.....

Solution 4.

(a) Using the usual steps of separating and integrating, we find

$$\begin{aligned}
 yy' &= \frac{x^2}{1+x^3} \\
 ydy &= \frac{x^2}{1+x^3} dx \\
 \int ydy &= \int \frac{x^2}{1+x^3} dx \\
 \frac{1}{2}y^2 &= \frac{1}{3} \log |1+x^3| + C \\
 y^2 &= \frac{2}{3} \log |1+x^3| + C \\
 y &= \pm \sqrt{\frac{2}{3} \log |1+x^3| + C}
 \end{aligned}$$

(b) Using the usual steps of separating and integrating, we find

$$\begin{aligned}
 y' &= y^2 \sin(x) \\
 \frac{1}{y^2} y' &= \sin(x) \\
 \frac{1}{y^2} dy &= \sin(x) dx \\
 \int \frac{1}{y^2} dy &= \int \sin(x) dx \\
 -\frac{1}{y} &= -\cos(x) + C \\
 y &= \frac{1}{\cos(x) + C}
 \end{aligned}$$

(c) Using the usual steps of separating and integrating, we find

$$\begin{aligned}(y + e^y)\frac{dy}{dx} &= x - e^{-x} \\ (y + e^y)dy &= (x - e^{-x})dx \\ \int (y + e^y)dy &= \int (x - e^{-x})dx \\ \frac{1}{2}y^2 + e^y &= \frac{1}{2}x + e^{-x} + C\end{aligned}$$

This equation is too hard to solve for y , so we leave it in this form.

(d) Using the usual steps of separating and integrating, we find

$$\begin{aligned}(1 + y^2)\frac{dy}{dx} &= x^2 \\ (1 + y^2)dy &= x^2dx \\ \int (1 + y^2)dy &= \int x^2dx \\ y + \frac{1}{3}y^3 &= \frac{1}{3}x^3 + C\end{aligned}$$

This equation is too hard to solve for y , so we leave it in this form.

Problem 5 *Separable Initial Value Problems*

For each of the following initial value problems

(i) Solve the initial value problem

(ii) Using a computer, graph the solution

Attach a printout of your graph to your homework

(iii) Determine as accurately as you can the interval in which the solution is defined

(a) $x dx + ye^{-x} dy = 0$, $y(0) = 1$

(b) $y' = \frac{3x^2 - e^x}{2y - 5}$, $y(0) = 1$

.....

Solution 5.

(a) We start out by finding a big family of solutions

$$\begin{aligned} xdx &= -ye^{-x}dy \\ xe^x dx &= -ydy \\ \int xe^x dx &= \int -ydy \\ xe^x - e^x + C &= -\frac{1}{2}y^2 \\ y &= \pm\sqrt{-2xe^x + 2e^x + C} \end{aligned}$$

Now from the initial condition, we have $y(0) = 1$, so we know that we want the + in front of the square root above. Also this means that

$$1 = \sqrt{-2(0)e^0 + 2e^0 + C},$$

and therefore $C = -1$. We conclude

$$y = \sqrt{-2xe^x + 2e^x - 1}$$

This solution is defined for $-2xe^x + 2e^x - 1 \geq 0$, or for x between approximately -1.678 and 0.768 .

(b) We start out by finding a big family of solutions

$$\begin{aligned} (2y - 5)y' &= 3x^2 - e^x \\ (2y - 5)dy &= (3x^2 - e^x)dx \\ \int (2y - 5)dy &= \int (3x^2 - e^x)dx \\ y^2 - 5y &= x^3 - e^x + C \\ y &= \frac{5 \pm \sqrt{25 + 4(x^3 - e^x + C)}}{2} \end{aligned}$$

Now $y(0) = 1$ tells us that we want the negative sign in front of the square root! Also it says

$$1 = \frac{5 - \sqrt{25 + 4C}}{2}$$

and therefore $C = -3$. Hence the solution is

$$y = \frac{5 - \sqrt{25 + 4x^3 - 4e^x - 12}}{2}$$

This is defined for $25 + 4x^3 - 4e^x - 12 \geq 0$ or x between approximately -1.4445 and 4.6297 .

Problem 6 *Homogeneous Equations*

For each of the following, show that the equation is homogeneous. Then find a family of solutions differing by a constant

(a) $\frac{dy}{dx} = \frac{3y^2 - x^2}{2xy}$

(b) $\frac{dy}{dx} = -\frac{4x + 3y}{2x + y}$

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Solution 6.

(a) Since

$$\frac{dy}{dx} = \frac{3}{2}(y/x) - \frac{1}{2}(x/y)$$

it is clear that this differential equation is homogeneous. By setting $z = y/x$, we know that $y = xz$, and therefore

$$\frac{dy}{dx} = z + x \frac{dz}{dx}.$$

Thus the old differential equation may be rewritten as

$$z + x \frac{dz}{dx} = \frac{3}{2}z - \frac{1}{2} \frac{1}{z}.$$

This is separable! We then solve for z in the usual way

$$\begin{aligned} x \frac{dz}{dx} &= \frac{1}{2}z - \frac{1}{2} \frac{1}{z} \\ x \frac{dz}{dx} &= \frac{1}{2} \frac{z^2 - 1}{z} \\ 2 \frac{z}{z^2 - 1} \frac{dz}{dx} &= \frac{1}{x} \\ 2 \frac{z}{z^2 - 1} dz &= \frac{1}{x} dx \\ \int 2 \frac{z}{z^2 - 1} dz &= \int \frac{1}{x} dx \\ \ln |z^2 - 1| &= \ln |x| + C \\ z^2 - 1 &= Cx \\ z^2 &= Cx + 1 \\ z &= \pm \sqrt{Cx + 1} \end{aligned}$$

Since $y = xz$, it then follows

$$y = \pm x \sqrt{Cx + 1}.$$

(b) Since

$$\frac{dy}{dx} = -\frac{4 + 3y/x}{2 + y/x}$$

it is clear that this differential equation is homogeneous. By setting $z = y/x$, we know that $y = xz$ and therefore

$$\frac{dy}{dx} = z + x \frac{dz}{dx}.$$

Thus the old differential equation may be rewritten as

$$z + x \frac{dz}{dx} = -\frac{4 + 3z}{2 + z}$$

This is separable! We then solve for z in the usual way

$$\begin{aligned} x \frac{dz}{dx} &= -\frac{4 + 3z}{2 + z} - z \\ x \frac{dz}{dx} &= -\frac{4 + 5z + z^2}{2 + z} \\ -\frac{2 + z}{4 + 5z + z^2} \frac{dz}{dx} &= \frac{1}{x} \\ -\left(\frac{1}{3} \frac{1}{z + 1} + \frac{2}{3} \frac{1}{z + 4}\right) \frac{dz}{dx} &= \frac{1}{x} \\ -\left(\frac{1}{3} \frac{1}{z + 1} + \frac{2}{3} \frac{1}{z + 4}\right) dz &= \frac{1}{x} dx \\ -\int \left(\frac{1}{3} \frac{1}{z + 1} + \frac{2}{3} \frac{1}{z + 4}\right) dz &= \int \frac{1}{x} dx \\ -\left(\frac{1}{3} \log |z + 1| + \frac{2}{3} \log |z + 4|\right) &= \log |x| + C \\ \log |z + 1| + 2 \log |z + 4| &= -3 \log |x| + C \\ \log |z + 1| + \log (z + 4)^2 &= -3 \log |x| + C \\ \log |(z + 1)(z + 4)^2| &= -3 \log |x| + C \\ (z + 1)(z + 4)^2 &= Cx^{-3} \end{aligned}$$

This is as far as we can reasonably simplify. Now using the fact that $z = y/x$, we get the equation

$$\left(\frac{y}{x} + 1\right) \left(\frac{y}{x} + 4\right)^2 = Cx^{-3}$$

Multiplying both sides by x^3 makes this look a bit better

$$(y + x)(y + 4x)^2 = C.$$