Math 307 Lecture 4

First Order Linear Equations!

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Today!

Last time:

- First-Order Linear Equations
- Method: Integrating factors
- Method: Variation of parameters

This time:

- More practice with First-Order Linear and Separable ODEs

Next time:

- Modeling with First Order Equations
- First homework due Friday!!!

Today's lecture brought to you by

Outline

Solve the separable equation:

Example

Solve the separable equation $y' = \cos^2(x)\cos^2(2y)$

First we "separate":

$$\sec^2(2y)y' = \cos^2(x)$$

$$\sec^2(2y)dy = \cos^2(x)dx$$

Then we integrate:

$$\int \sec^2(2y)dy = \int \cos^2(x)dx$$

$$\frac{1}{2}\tan(2y) = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$$

Lastly, if we can, solve for y:

$$y = \frac{1}{2}\arctan(x + \frac{1}{2}\sin(2x) + C)$$

Solve the homogeneous equation:

Example

Solve the homogeneous equation $y' = \frac{x+3y}{x-y}$

Put it in a homogeneous form:

$$y' = \frac{1+3(y/x)}{1-(y/x)}$$

Use the change of variables

$$z = y/x, y' = z + xz'$$

$$z + xz' = \frac{1+3z}{1-z}$$

Separate:

$$xz' = \frac{1+2z+z^2}{1-z}$$

$$\frac{1-z}{1+2z+z^2}dz = \frac{1}{x}dx$$

Integrate:

$$\int \frac{1-z}{1+2z+z^2}dz = \int \frac{1}{x}dx$$

Solve the homogeneous equation (continued):

Example

Solve the homogeneous equation $y' = \frac{x+3y}{x-y}$

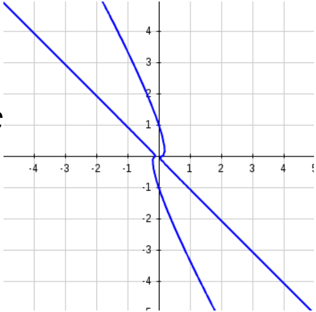
$$\frac{-2}{1+z} - \ln|1+z| = \ln|x| + C$$

Replace $z = y/x$:

$$\frac{-2}{1+(y/x)} - \ln|1+(y/x)| = \ln|x| + C$$

Hard to solve for y (use Weierstrass W-function)

Figure : Plot of curve for $c = 0$



Solve with an integrating factor:

Example

Solve the first-order linear equation $y' + 2ty = 2te^{-t^2}$ by finding an integrating factor.

- Suppose that $\mu = \mu(t)$ is the integrating factor
- Then the linear equation

$$\mu(t)y' + 2t\mu(t)y = 2t\mu(t)e^{-t^2}$$

  must be exact!
- This means $\mu'(t) = 2t\mu(t)$.

Solve with an integrating factor (continued):

Example

Solve the first-order linear equation $y' + 2ty = 2te^{-t^2}$ by finding an integrating factor.

- The equation $\mu'(t) = 2t\mu(t)$ is separable

$$2tdt = \frac{1}{\mu}d\mu$$

$$\int 2tdt = \int \frac{1}{\mu}d\mu$$

$$t^2 + C = \ln|\mu|$$

Need only 1 solution ($C = 0$)

$$\mu = e^{t^2}$$

Solve with an integrating factor (continued):

Example

Solve the first-order linear equation $y' + 2ty = 2te^{-t^2}$ by finding an integrating factor.

Plug in μ to get an exact equation

$$e^{t^2}y' + 2te^{t^2}y = 2t$$

Gather up the y parts

$$(e^{t^2}y)' = 2t$$

Then we integrate:

$$\int (e^{t^2}y)'dt = \int 2tdt$$

$$e^{t^2}y = t^2 + C$$

Solve for y

$$y = t^2e^{-t^2} + Ce^{-t^2}$$

Integrating factors, another example!

Example

Solve the first-order linear equation $y' + y = 5\sin(2t)$ by finding an integrating factor.

- Let $\mu = \mu(t)$ be the integrating factor
- Then the linear equation

$$\mu(t)y' + \mu(t)y = 5\mu(t)\sin(2t)$$

  must be exact!
- This means that

$$\mu'(t) = \mu(t)$$

Solve with an integrating factor (continued):

Example

Solve the first-order linear equation $y' + y = 5\sin(2t)$ by finding an integrating factor.

- The equation $\mu(t) = \mu'(t)$ is separable!

$$dt = \frac{1}{\mu} d\mu$$

$$\int dt = \int \frac{1}{\mu} d\mu$$

$$t + C = \ln|\mu|$$

Need only 1 solution ($C = 0$)

$$\mu = e^t$$

Solve with an integrating factor (continued):

Example

Solve the first-order linear equation $y' + y = 5\sin(2t)$ by finding an integrating factor.

Plug in μ to get an exact equation

$$e^t y' + e^t y = 5e^t \sin(2t)$$

Gather up the y parts

$$(e^t y)' = 5e^t \sin(2t)$$

Then we integrate:

$$\int (e^t y)' dt = \int 5e^t \sin(2t) dt$$

$$e^t y = e^t \sin(2t) - 2e^t \cos(2t) + C$$

Solve for y

$$y = \sin(2t) - 2\cos(2t) + Ce^{-t}$$

Solve with variation of parameters:

Example

Solve the first-order linear equation $ty' + 2y = \sin(t)$ by variation of parameters.

First solve the homogeneous equation:

$$ty_h' + 2y_h = 0$$

This is separable!

$$ty_h' = -2y_h$$

A solution is $y_h = t^{-2}$

Then define $v(t)$ by $y = vy_h$
Then from V.O.P. equation

$$v(t) = \int \frac{q(t)}{y_h(t)}dt$$

with $q(t) = \sin(t)$, and thus

$$v(t) = 2t\sin(t) - (t^2 - 2)\cos(t) + C$$

Lastly $y = vy_h$

$$y = v(t)y_h = 2t^{-1}\sin(t) - (1 - 2t^{-2})\cos(t) + Ct^{-2}$$

Summary!

What we did today:

- We reviewed our current toolbox of solution methods

Plan for next time:

- First homework due Friday!!
- Modeling with first order equations