Math 307 Quiz 2

October 6, 2014

Problem 1. Explain why the following first order linear equation is exact. Then determine its general solution

$$te^t y' = -(1+t)e^t y + e^{2t}$$

Solution 1. In a, b, c form, this equation is

$$\overbrace{te^t}^{a(t)} y' + \overbrace{(1+t)e^t}^{b(t)} y = \overbrace{e^{2t}}^{c(t)}.$$

Then since

$$a'(t) = (te^t)' = e^t + te^t = (1+t)e^t = b(t),$$

we see that this equation is exact. To solve it, we regroup the terms on the left-hand side:

$$(te^t y)' = e^{2t},$$

and then integrate both sides with respect to t

$$\begin{aligned}\int (te^t y)' dt &= \int e^{2t} dt \\ te^t y &= \frac{1}{2}e^{2t} + C \\ y &= \frac{1}{2t}e^t + \frac{C}{t}e^{-t}.\end{aligned}$$

Problem 2. Determine an integrating factor for the equation

$$y' + \cot(t) y = e^t$$

Solution 2. We first put our equation in p, q form:

$$y' = \overbrace{-\cot(t)}^{p(t)}\, y + \overbrace{e^t}^{q(t)}.$$

Then we know that the integrating factor is given by

$$\mu(t) = e^{-\int p(t)dt} = e^{\int \cot(t)dt} = e^{\ln(\sin(t))} = \sin(t).$$

Therefore the integrating factor is $\mu(t) = \sin(t)$.

Problem 3. Find a solution to the following initial value problem

$$y' + \cot(t)y = e^t, \quad y(\pi/2) = 1.$$

Solution 3. For this problem, if we were able to find an integrating factor in the last problem, we might as well use it. If not, we might consider instead using variation of parameters. We will use the integrating factor that we found in the previous problem.

Multiplying both sides of the equation by this integrating factor, we obtain the exact equation:

$$\sin(t)y' + \cos(t)y = e^t \sin(t).$$

It might be a good double-check at this point to make sure that the equation we obtained is, in fact, exact (this double checks that the integrating factor we found was correct). In this case, we see it is exact, since $\sin(t)' = \cos(t)$. Now we regroup the terms on the left-hand side:

$$(\sin(t)y)' = e^t \sin(t).$$

Finally, we integrate both sides with respect to t:

$$\begin{aligned}
\int (\sin(t)y)'dt &= \int e^t \sin(t)dt \\
\sin(t)y &= -\frac{1}{2}e^t \cos(t) + \frac{1}{2}e^t \sin(t) + C \\
y &= -\frac{1}{2}e^t \cot(t) + \frac{1}{2}e^t + C\csc(t).
\end{aligned}$$

Lastly, we must satisfy the initial condition that $y(\pi/2) = 1$. This means

$$1 = 0 + \frac{1}{2}e^{\pi/2} + C.$$

Therefore $C = 1 - \frac{1}{2}e^{\pi/2}$, and thus

$$y = -\frac{1}{2}e^t \cot(t) + \frac{1}{2}e^t + \left(1 - \frac{1}{2}e^{\pi/2}\right) \csc(t).$$