Math 307 Lecture 6

Differences Between Linear and Nonlinear Equations

W.R. Casper

Department of Mathematics
University of Washington

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Today!

Last time:

- Modeling first-order equations

This time:

- Differences between linear and nonlinear equations

Next time:

- Autonomous Equations and Population Dynamics

Outline

1. Existence and Uniqueness for First Order Linear Equations
   - First Order Linear Equations
   - The Theorem
2. Existence and Uniqueness for General First Order Equations
   - The Nonlinear Case
   - The Theorem
3. Differences between Linear and Nonlinear
   - Virtues of Linear Equations

When do solutions to IVPs exist?

- Consider an arbitrary first order linear IVP

$$y' = p(t)y + q(t), \quad y(t_0) = y_0$$

- When do we know if a solution exists?
- When do we know if the solution is unique?
- Not all the time!

Example

The initial value problem

$$y' = \frac{1}{x}, \quad y(0) = 0$$

does not have a solution.

What went wrong?

- Why didn't our example have a solution?
- Because the function $1/x$ isn't continuous at 0
- Consider instead a slightly different example

Example

The initial value problem

$$y' = \frac{1}{x}, \quad y(1) = 0$$

has the solution $y(t) = \ln(t)$ defined.

Existence/Uniqueness Theorem

- What's the largest interval where this is defined?
- It's defined on $(0, \infty)$, the same interval where $1/x$ is
- Is the solution of this IVP unique?
- Yes! Think fundamental theorem of calculus

Theorem

If p, q are continuous functions on an interval $I = (a, b)$ and $t_0 \in I$, then the initial value problem

$$y' = p(t)y + q(t), \quad y(t_0) = y_0$$

has a unique solution defined on I.

An Example

Example

What is the largest interval on which the initial value problem

$$\sin(t)y' = -\cos(t)y + 2t, \quad y(\pi/2) = 1.$$

has a unique solution?

- Dividing by $\sin(t)$, we get $y' = p(t)y + q(t)$, where $p(t) = -\cot(t)$ and $q(t) = t\csc(t)$.
- $\csc(t)$ and $\cot(t)$ has discontinuities at 0 and π
- Unique solution is guaranteed to be defined on the interval $(0, \pi)$
- In fact, solution is $y(t) = t^2\csc(t) + \left(1 - \frac{\pi^2}{4}\right)\csc(t)$

When do solutions to IVPs exist?

Example

Consider the initial value problem

$$y' = y^{1/3}, \quad y(0) = 0$$

- the above differential equation is nonlinear (why?).
- It also has more than one solution!
- For any choice of $\ell \geq 0$

$$y(t) = \begin{cases} 0, & 0 \leq t < \ell \\ \pm[\frac{2}{3}(t-\ell)]^{3/2}, & t \geq \ell \end{cases}$$

is a solution

What went wrong?

- Why did our example not have a unique solution?
- If f isn't linear, it'd better have some other constraint!
- Consider instead a slightly different example

Example

Solve the initial value problem

$$y' = y^{1/3}, \quad y(1) = 1$$

- has a unique solution!

$$y(t) = \left(\frac{2}{3}t + \frac{1}{3}\right)^{3/2}$$

Existence/Uniqueness Theorem

- What changed the second time?
- Both f and $\partial f/\partial y$ are continuous in an open rectangle about $(1, 1)$

Theorem

If f and $\frac{\partial f}{\partial y}$ are continuous in an open rectangle about (t_0, y_0), then the initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0$$

has a unique solution on some open interval containing t_0.

An Example

Example

Consider the initial value problem

$$y' = y^2, \;\; y(0) = 1.$$

In what interval does a solution exist?

- From our previous theorem, we know that a solution exists in *some* interval
- Since it's separable, we can actually solve it

$$y = \frac{1}{1-t}$$

- The solution is defined in the interval $(-\infty, 1)$
- Tough to determine interval without actually solving it

Nice Properties of Linear Equations

- general solutions exist under simple assumptions
- general solution is always of the form
  $y = \text{function} + C\text{function}$
- easy to determine the interval on which a solution is defined
- "superposition principal" helps us build new solutions from old ones

The Downside for Nonlinear Equations

- the required assumptions are more complicated for existence/uniqueness
- general solution can be more complicated than something involving a single constant
- harder to determine the interval on which a solution exists

Upshot for Nonlinear Equations

- mathematically much more interesting
- much richer behavior
- can be aperiodic, chaotic, and more!

Review!

Today:

- Differences between linear and nonlinear equations
- Existence and uniqueness theorems

Next time:

- Autonomous Equations and Population Dynamics