Math 307 Lecture 10

Second-Order Homogeneous Linear ODEs with Constant Coefficients

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Today!

Last time:

- Return of Exact Equations and Integrating Factors!

This time:

- Higher-order Homogeneous Linear Equations with Constant Coefficients

Next time:

- Review for First Midterm

Outline

Second Order Linear Equations

- What is a linear ordinary differential equation of second order?

Definition

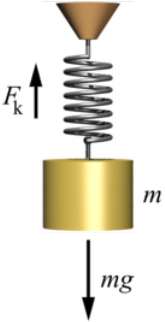
A second-order linear equation is an equation of the form

$$y'' + p(t)y' + q(t)y = f(t)$$

- For reasons that stem from physics, the function f is called the *forcing function*

Example: An Ideal Mass-Spring System

Figure : A mass spring system can be modeled by a second-order equation



- Let x be effective length of spring (length - equilibrium length)
- Spring force: $F_{\text{spring}} = -kx$ with k the spring constant
- Grav force: $F_{\text{grav}} = mg$
- $F_{\text{total}} = F_{\text{spring}} + F_{\text{grav}}$
- Newton's Law: $F = mx''$
- Hence motion satisfies a second-order linear ODE

$$mx'' = -kx + mg = 0$$

Second-Order Linear Homogeneous Equations

- When is a Second-Order Linear Equation Homogeneous?

Definition

A second-order linear equation

$$y'' + p(t)y' + q(t)y = f(t)$$

is called homogeneous if $f(t) = 0$.

- For example:
- homogeneous: $y'' + xy' + \sin(x)y = 0$
- NOT homogeneous: $y'' - \sec(x)y = x$

The Superposition Principle

- Homogeneous linear equations are nice, because they satisfy the “superposition principle":

Theorem (Superposition Principle)

If y_1 and y_2 are two solutions to a homogeneous linear ODE, then for any constants c_1 and c_2, the function $c_1 y_1 + c_2 y_2$ is also a solution

- For example:
- $\sin(x)$ and $\cos(x)$ are two solutions of the homogeneous linear ODE $y'' + y = 0$ (check this!!)
- By the superposition principal, $-13\sin(t) + 2\cos(t)$ is also a solution to the ODE (double check!!)

Linear Equations with Constant Coefficients

- What does it mean for a second order linear ODE to have constant coefficients?

Definition

A second-order linear equation

$$y'' + p(t)y' + q(t)y = f(t)$$

has constant coefficients if p and q are constant.

- For example:
- $y'' + xy' + \sin(x)y = 0$ does NOT have const. coeff.
- $y'' + 3y' - 12y = \sec(x)$ does have const. coeff.

How to Solve Hom. Linear ODEs with Const. Coeff

Figure : A Group of Dogs Solving ODEs by bluffing

- How do we solve second-order linear homogeneous ODEs with constant coefficients?
- We make a **crazy bluff** – that we already know the actual solution!
- We say that the solution is

$$y = e^{rt}$$

- How crazy is that?

How to Solve Hom. Linear ODEs with Const. Coeff

- But it works! If $y = e^{rt}$, then

$$y' = re^{rt}$$
$$y'' = r^2 e^{rt}$$

- Since y is a solution of our ODE, this means

$$\begin{aligned} 0 &= y'' + ay' + by \\ &= r^2 e^{rt} + are^{rt} + be^{rt} \\ &= (r^2 + ar + b)e^{rt} \end{aligned}$$

- Hence r must be a root of the polynomial $x^2 + ax + b$.

How to Solve Hom. Linear ODEs with Const. Coeff

- The polynomial $x^2 + ax + b$ is called the *characteristic polynomial* of the equation
- The polynomial $x^2 + ax + b$ will have two roots, say r_1 and r_2
- If the roots are **distinct**, then we will have **two** solutions $e^{r_1 t}$ and $e^{r_2 t}$
- By the **superposition principal**, we know

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

  will be a solution as well!
- In fact, this will be the general solution

Our First Example

Example

Find the general solution of the second-order homogeneous linear ODE

$$y'' + 2y' - 3y = 0$$

- How do we find the general solution?
- First we bluff and say we already know our solution: $y = e^{rt}$.
- This means $y' = re^{rt}$ and $y'' = r^2 e^{rt}$

Our First Example

- Then since y is a solution

$$\begin{aligned} 0 &= y'' + 2y' - 3y \\ &= r^2 e^{rt} + 2re^{rt} - 3e^{rt} \\ &= (r^2 + 2r - 3)e^{rt} \end{aligned}$$

- Therefore $r^2 + 2r - 3 = 0$
- Factoring: $(r + 3)(r - 1) = 0$, so $r = -3$ or $r = 1$
- This means $y = e^{-3t}$ and $y = e^t$ are both solutions!
- General solution:

$$y = Ae^{-3t} + Be^t, \qquad A, B \text{ constants}$$

A Second Example

Example

Find the general solution of the second-order homogeneous linear ODE

$$6y'' - y' - y = 0$$

- How do we find the general solution?
- First we bluff and say we already know our solution: $y = e^{rt}$.
- This means $y' = re^{rt}$ and $y'' = r^2e^{rt}$

A Second Example

- Then since y is a solution

$$\begin{aligned} 0 &= 6y'' - y' - y \\ &= 6r^2e^{rt} - re^{rt} - e^{rt} \\ &= (6r^2 - r - 1)e^{rt} \end{aligned}$$

- Therefore $6r^2 - r - 1 = 0$
- Factoring: $(2r - 1)(3r + 1) = 0$, so $r = 1/2$ or $r = -1/3$
- This means $y = e^{t/2}$ and $y = e^{-t/3}$ are both solutions!
- General solution:

$$y = Ae^{-3t} + Be^{t}, \qquad A, B \text{ constants}$$

IVP's for Second-Order Equations

- When it comes to second order linear equations, we've also got initial value problems
- However, since there are two arbitrary constants in our general solution, we need more information to specify the "initial condition"
- At the initial time t_0, we need to specify the initial value $y(t_0)$ and initial first derivative $y'(t_0)$
- Given the data $y(t_0) = y_0$ and $y'(t_0) = y'_0$, we get a unique solution to the IVP!

IVP Example

Example

Find a solution to the IVP

$$y'' + y' - 2y = 0, \quad y(0) = 1, \; y'(0) = 1$$

- We propose a solution of the form $y = e^{rt}$.
- Then $y'' + y' - 2y = 0$ gives us the *characteristic equation*

$$r^2 + r - 2 = 0$$

- Factoring, we get $(r+2)(r-1) = 0$, so the general solution is

$$y = Ae^{-2t} + Be^{t}.$$

IVP Example

- Now notice that
$$y(0) = A + B$$
- and also
$$y'(0) = -2A + B$$
- So the conditions $y(0) = 1$ and $y'(0) = 1$ tell us
$$A + B = 1$$
$$-2A + B = 1$$
- The solution is $A = 0$ and $B = 1$, so the final answer is
$$y(t) = e^t$$

Practice

Try the following practice problems:

- Find the general solution of the IVP

$$y'' + 5y' = 0$$

- Find the solution of the IVP

$$6y'' - 5y + y = 0, \quad y(0) = 4, y'(0) = 0$$

Review!

Today:

- Higher-order Homogeneous Linear Equations with Constant Coefficients

Next time:

- Review for the first exam