Math 307 Lecture 9

Exact Equations and Integrating Factors Part II

W.R. Casper

Department of Mathematics
University of Washington

October 31, 2014

Today!

Last time:

- Numerical Approximations

This time:

- Return of Exact Equations and Integrating Factors!

Next time:

- Higher-order Homogeneous Linear Equations with Constant Coefficients

Outline

What is an Exact Equation?

- Remember that any first-order ODE can be written in the form

$$M(x, y) + N(x, y)y' = 0.$$

- Recall the definition of an exact equation:

Definition

An equation of the above form is *exact* if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

- That's well and good, but let's see some examples!

What is an Exact Equation?

In each of the following, let's try to decide if the equation is exact

- $y\cos(xy) + x\cos(xy)y' = 0$
- Yes!
- $y\cos(xy) - \tan(x) + x\cos(xy)y' = 0$
- Yuppers!
- $2xy + x^2yy' = 0$
- No sir!
- $ye^{xy} + (xe^{xy} + \sec^2(y))y' = 0$
- You 'betcha!

Why are Exact Equations Nice?

- The next theorem tells us why we **love** exact equations:

Theorem

Suppose that the equation

$$M(x, y) + N(x, y)y' = 0$$

is exact and that M, N are “nice enough". Then there exists a function $\psi(x, y)$ such that

$$\frac{\partial\psi}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial\psi}{\partial y} = N(x, y)$$

Why are Exact Equations Nice?

- It may not look like it yet, but the previous theorem helps us to solve exact equations!
- To see why, let's calculate $\frac{d\psi}{dx}$ using implicit differentiation:

$$\begin{aligned}\frac{d\psi(x,y)}{dx} &= \frac{\partial\psi}{\partial x}\frac{dx}{dx} + \frac{\partial\psi}{\partial y}\frac{dy}{dx}\\ &= \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}y'\\ &= M(x,y) + N(x,y)y'\end{aligned}$$

Why are Exact Equations Nice?

- Thus the differential equation

$$M(x,y) + N(x,y)y' = 0$$

  becomes the equation

$$\frac{d\psi(x,y)}{dx} = 0$$

- Integrating both sides with respect to x, the solution is then

$$\psi(x,y) = C$$

- We've solved the exact equation!!

Solving Exact Equations

- We've found that the solution to an exact equation is given implicitly by $\psi(x, y) = C$
- One important question you should be asking yourself now is how do we determine $\psi(x, y)$?
- The answer is "partial integration"
- To see what we mean, let's look at some examples!

Solving Exact Equations: A First Example

- Consider the equation

$$\overbrace{y\cos(xy) - \tan(x)}^{M(x,y)} + \overbrace{x\cos(xy)}^{N(x,y)}\, y' = 0$$

- This equation is exact (check this!)
- This means that there is a $\psi(x, y)$ satisfying

$$\frac{\partial\psi(x, y)}{\partial y} = N(x, y) = x\cos(xy).$$

- Doing a partial integral of both sides,

$$\int \frac{\partial\psi(x, y)}{\partial y}\partial y = \int x\cos(xy)\partial y$$

Solving Exact Equations: A First Example

- Doing a partial integral of both sides,

$$\psi(x, y) = \int x \cos(xy) \partial y$$

- To do the partial integral wrt. y, you treat x as a constant:

$$\psi(x, y) = \overbrace{\sin(xy)}^{\text{integral with } x \text{ constant}} + \overbrace{g(x)}^{\text{arbit. func. of integ.}}$$

- With partial integrals, we end up with an arbitrary “function of integration" instead of an arbitrary constant
- Notice we integrated wrt. y so we get an arbitrary func of x

Solving Exact Equations: A First Example

- How can we figure out what $g(x)$ must be?
- Remember that

$$\partial\psi/\partial x = M(x, y)$$

- Therefore

$$y\cos(xy) + g'(x) = y\cos(xy) - \tan(x).$$

- This simplifies to $g'(x) = -\tan(x)$, so that $g(x) = \ln|\cos(x)|$ (we can forget the constant)
- Thus

$$\psi(x, y) = \sin(xy) + \ln|\cos(x)|$$

- Solution is therefore $\sin(xy) + \ln|\cos(x)| = C$

Solving Exact Equations: A Second Example

- Let's look at another example!
- Consider the equation

$$\overbrace{ye^{xy}}^{M(x,y)} + \overbrace{xe^{xy} + \sec^2(y)}^{N(x,y)}\, y' = 0$$

- This equation is exact (check this!)
- This means that there is a $\psi(x, y)$ satisfying

$$\frac{\partial\psi(x,y)}{\partial x} = M(x,y) = ye^{xy}.$$

- Doing a partial integral of both sides,

$$\int \frac{\partial\psi(x,y)}{\partial x}\partial x = \int ye^{xy}\partial x$$

Solving Exact Equations: A Second Example

- Doing a partial integral of both sides,

$$\psi(x, y) = \int ye^{xy}\partial x$$

- To do the partial integral wrt. x, you treat y as a constant:

$$\psi(x, y) = e^{xy} + h(y)$$

- With partial integrals, we end up with an arbitrary “function of integration" $h(y)$ instead of an arbitrary constant
- Notice we integrated wrt. x so we get an arbitrary func of y

Solving Exact Equations: A Second Example

- How can we figure out what $h(y)$ must be?
- Remember that

$$\partial\psi/\partial y = N(x, y)$$

- Therefore

$$xe^{xy} + h'(y) = xe^{xy} + \sec^2(y).$$

- This simplifies to $h'(y) = \sec^2(y)$, so that $h(y) = \tan(y)$ (we can forget the constant)
- Thus

$$\psi(x, y) = e^{xy} + \tan(y)$$

- Solution is therefore $e^{xy} + \tan(y) = C$

What if the Equation is not Exact?

Figure : A graphical depiction of the ecstasy one feels when the nonlinear first order equation is exact

What if the Equation is not Exact?

Figure : Eeyore never gets exact equations on exams

- "most" nonlinear first order equations are not exact
- Q: what should we do with such an equation?
- A: try to find an integrating factor!
- this will be **totally impossible** in general
- but it will work often enough to make it worth a try...

What if the Equation is not Exact?

- Remember, an integrating factor is a function $\mu(x, y)$ that we multiply by to make the equation exact
- When we try to find one, we often make an assumption about the form
- eg. $\mu(x, y) = \mu(x)$
- or $\mu(x, y) = \mu(y)$
- or $\mu(x, y) = x^a y^b$
- these might not work; μ may be too hard to find!

Figure : Eeyore getting help from his turtle friend to find an integrating factor

Integrating Factor Example 1

Example

Solve the first order equation

$$y + (2x - ye^y)y' = 0$$

by finding an integrating factor of the form $\mu(x, y) = \mu(y)$.

- $\mu(y)y + (2x - ye^y)\mu(y)y' = 0$ exact
- Implies $\mu'(y)y + \mu(y) = 2\mu(y)$
- Results in $\mu'(y)y = \mu(y)$; a solution is $\mu(y) = y$

Integrating Factor Example 1

- Thus we get an exact equation

$$y^2 + (2xy - y^2e^y)y' = 0$$

- We know $\psi(x, y) = \int y^2 \partial x = xy^2 + h(y)$
- Since $\frac{\partial\psi}{\partial y} = N(x, y)$, we also know $2xy + h'(y) = 2xy - y^2e^y$
- Hence $h = \int -y^2e^y dy = -(y^2 - 2y + 2)e^y$
- Thus $\psi(x, y) = xy^2 - (y^2 - 2y + 2)e^y$
- Solution is $\psi(x, y) = C$, ie.

$$xy^2 - (y^2 - 2y + 2)e^y = C$$

Integrating Factor Example 2

Example

Solve the first order equation

$$(x+2)\sin(y) + x\cos(y)y' = 0$$

by finding an integrating factor of the form $\mu(x, y) = \mu(x)$.

- Try it yourself!

Integrating Factor Example 3

Example

Solve the first order equation

$$x^2y^3 + x(1+y^2)y' = 0$$

by finding an integrating factor of the form $\mu(x,y) = x^a y^b$ for some constants a, b

- Try it yourself!

Review!

Today:

- More on Exact Equations and Integrating Factors

Next time:

- Higher-order Homogeneous Linear Equations with Constant Coefficients