Math 307 Quiz 5

October 20, 2014

Problem 1. Determine if the following equation is exact. If it is, find the general solution.

$$3x^2y^4 + \cos(x) + (4x^3y^3 + 2y)y' = 0$$

Solution 1. We calculate

$$M_y = 12x^2y^3$$

and

$$N_x = 12x^2y^3$$

so therefore the equation is exact. Thus there must exist a function $\psi(x, y)$ satisfying $\psi_x = M$ and $\psi_y = N$. This means that

$$\begin{aligned}
\psi(x, y) &= \int \psi_x \partial x \\
&= \int M(x, y) \partial x \\
&= \int (3x^2y^4 + \cos(x)) \partial x \\
&= x^3y^4 + \sin(x) + g(y).
\end{aligned}$$

Furthermore, we know that

$$\psi_y = N(x, y) = 4x^3y^3 + 2y$$

and since $\psi = x^3y^4 + \sin(x) + g(y)$, we calculate

$$\psi_y = 4x^3y^3 + 0 + g'(y)$$

. Thus

$$4x^3y^3 + g'(y) = 4x^3y^3 + 2y$$

and it follows that $g'(y) = 2y$, so we may take $g(y) = y^2$. Hence

$$\psi(x, y) = x^3y^4 + \sin(x) + y^2.$$

The solution is given by setting $\psi(x, y) = C$. Therefore the general solution is

$$x^3y^4 + \sin(x) + y^2 = C.$$

Problem 2. Determine if the following equation is exact. If it is, find the general solution.

$$xe^{xy} - ye^{xy}y' = 0$$

Solution 2. We calculate

$$M_y = x^2e^{xy}$$

and also that

$$N_x = -y^2e^{xy}$$

Therefore the equation is not exact.

Problem 3. Find an integrating factor for the following equation

$$\tan(y) + xy' = 0$$

You do NOT need to solve the equation.

Solution 3. We try an integrating factor of the form $\mu(x, y) = \mu(y)$. Then

$$\overbrace{\mu(y)\tan(y)}^{M(x,y)} + \overbrace{x\mu(y)}^{N(x,y)} y' = 0$$

must be exact. Therefore $M_y = N_x$, meaning

$$\mu'(y)\tan(y) + \mu(y)\sec^2(y) = \mu(y),$$

and therefore

$$\mu'(y)\tan(y) = \mu(y)(1 - \sec^2(y)).$$

Now using the identity $\tan^2(y) + 1 = \sec^2(y)$, we obtain

$$\mu'(y)\tan(y) = \mu(y)(-\tan^2(y)),$$

which is a separable equation. We thereby obtain

$$\int \frac{1}{\mu} d\mu = \int -\tan(y) dy$$

and therefore

$$\ln \mu = \ln \cos(y) + C.$$

Thus we find an integrating factor to be $\mu(y) = \cos(y)$.