

Math 307 Lecture 11

Second-Order Homogeneous Linear ODEs with Constant Coefficients II

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Today!

Last time:

- Looked at 2nd-Order Hom. Lin. Eqns. with Constant Coefficients for the first time
- Found out how to solve them in the case that the characteristic polynomial had distinct roots

This time:

- Complex numbers
- 2nd-Order Hom. Lin. Eqns. with Constant Coefficients with characteristic polynomials having complex roots

Next time:

- 2nd-Order Hom. Lin. Eqns. with Constant Coefficients with characteristic polynomials having repeated roots

Outline

- 1 **Complex Numbers**
 - Complex Number Basics
 - Euler's Definition

- 2 **Complex Roots of the Characteristic Polynomial**
 - General solutions to 2nd Order Linear ODEs with Const. Coeff
 - General Case

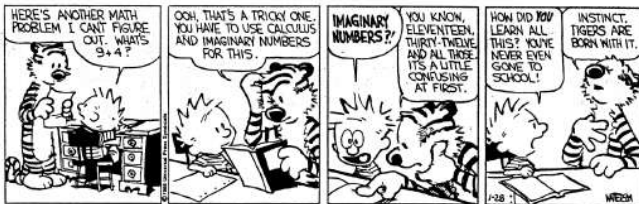
What are imaginary numbers?

- An *imaginary number* is any real number multiplied by $\sqrt{-1}$
- Usually denote this by i
- **examples:** $13i$, $\sqrt{2}i$, $-4i$, πi

Figure : Imaginary numbers are very real to tigers

Calvin and Hobbes

by Bill Watterson



What are complex numbers?

Definition

A *complex number* z is a number of the form

$$z = \underbrace{a}_{\text{real part}} + \underbrace{b}_{\text{imaginary part}} i$$

where a and b are any real numbers.

- Any real number is a complex number also
- Any imaginary number is a complex number also
- $2 + 3i$, $-\sqrt{7} - \frac{1}{2}i$ and $4 + 2\pi i$ are complex numbers too

Algebra with Complex numbers!

Question

How do we add, subtract, multiply, and divide complex numbers by other complex numbers?

Addition and Subtraction:

- To add complex things, just add real and imaginary parts.
- For example

$$(2 + 3i) + (4 + 2\pi i) = 6 + (3 + 2\pi)i$$

- Similar story for subtraction...

$$(2 + 3i) - (4 + 2\pi i) = -2 + (3 - 2\pi)i$$

Algebra with Complex numbers!

Question

How do we add, subtract, multiply, and divide complex numbers by other complex numbers?

Multiplication:

- To multiply complex things, we have to “foil”, remembering that $i^2 = -1$
- For example

$$\begin{aligned}(2 + 3i) \cdot (4 + 2\pi i) &= 2 \cdot 4 + 2 \cdot 2\pi i + 3i \cdot 4 + 3i \cdot 2\pi i \\ &= 8 + 4\pi i + 12i + 6\pi i^2 \\ &= 8 + 4\pi i + 12i - 6\pi \\ &= (8 - 6\pi) + (12 + 4\pi)i\end{aligned}$$

Algebra with Complex numbers!

Question

How do we add, subtract, multiply, and divide complex numbers by other complex numbers?

Inverses:

- We calculate the inverse of a complex number $z = a + bi$ by using the complex conjugate trick
- Complex conjugate is $z^* = a - bi$

$$\begin{aligned}\frac{1}{a + bi} &= \frac{1}{a + bi} \cdot 1 = \frac{1}{a + bi} \cdot \frac{(a - bi)}{(a - bi)} \\ &= \frac{a - bi}{(a + bi)(a - bi)} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i\end{aligned}$$

Algebra with Complex numbers!

Question

How do we add, subtract, multiply, and divide complex numbers by other complex numbers?

Division:

- We divide by multiplying by the inverse
- Alternatively we apply complex conjugate trick
- For example

$$\begin{aligned}\frac{4 + 2\pi i}{2 + 3i} &= \frac{4 + 2\pi i}{2 + 3i} \cdot 1 = \frac{4 + 2\pi i}{2 + 3i} \cdot \frac{(2 - 3i)}{(2 - 3i)} \\ &= \frac{(4 + 2\pi i) \cdot (2 - 3i)}{(2 + 3i) \cdot (2 - 3i)} = \frac{8 + 6\pi}{13} + \frac{4\pi - 12}{13}i\end{aligned}$$

Try it Yourself!

Have a go at the following calculations:



$$(-3 + 2i) + (4 - 6i) = ?$$



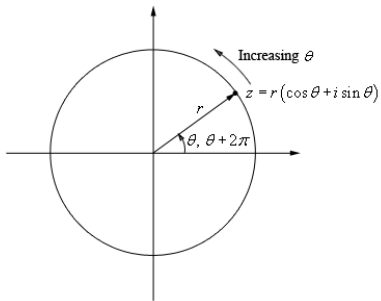
$$(3 + i) * (2 - i) = ?$$



$$\frac{1 + 2i}{3 - 7i} = ?$$

Complex numbers as vectors

Figure : A complex number is a vector in the plane



- We can “visualize” a complex number $z = x + iy$ as a *vector*
- The tip of the vector is put at the point (x, y)
- The base of the vector is put at the origin
- Using polar coordinates, we can write

$$x = r \cos(\theta)$$

$$y = r \sin(\theta)$$

Euler's Definition

Definition

If θ is a real number, then we *define*

$$e^{i\theta} = \cos(\theta) + i \sin(\theta).$$

Theorem

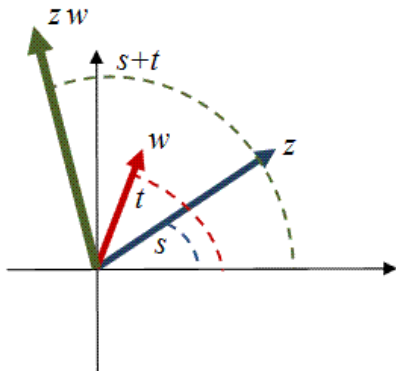
This definition does not break anything.

- Q: What do we mean by this?
- A: $e^{i\theta} e^{i\phi} = e^{i(\theta+\phi)}$
- Also, this makes sense in a power-series sort of way:

$$\sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!} i = \cos(\theta) + \sin(\theta)i = e^{i\theta}$$

Complex numbers as vectors

Figure : What does multiplication do to vectors?



- Take a complex number $x + iy$
- Convert (x, y) to polar:
 $r = \sqrt{x^2 + y^2}$
 $\theta = \tan^{-1}(y/x)$
- Then $x + iy = re^{i\theta}$ also!
- If $z = r_0 e^{is}$ and $w = r_1 e^{it}$ then

$$zw = r_0 r_1 e^{i(s+t)}$$

Angles add! (see figure)

A few more definitions

Figure : SMBC

Image redacted

- Let $z = re^{i\theta}$ be a complex number
- θ is called the *argument* of z
- r is called the *modulus* of z and is sometimes denoted as $|z|$
- easy exercise: show $r^2 = z \cdot z^*$
- Q: why do we care about complex numbers?
- A: simple! They give us more roots of polynomials!

Our First Example

Example

Find the general solution of the second-order homogeneous linear ODE

$$y'' - 2y' + 2y = 0$$

- How do we find the general solution?
- First we bluff and say we already know our solution:
 $y = e^{rt}$.
- This means $y' = re^{rt}$ and $y'' = r^2 e^{rt}$

Our First Example

- Then since y is a solution

$$\begin{aligned}0 &= y'' - 2y' + 2y \\ &= r^2 e^{rt} - 2r e^{rt} + 2e^{rt} \\ &= (r^2 - 2r + 2)e^{rt}\end{aligned}$$

- Therefore $r^2 - 2r + 2 = 0$
- Roots of the polynomial $r^2 - 2r + 2$ are $1 \pm i$
- This means $y = e^{(1+i)t}$ and $y = e^{(1-i)t}$ are both solutions!
- Wait a minute, these are complex-valued functions!

Our First Example

- We only want real-valued solutions. So look at the general solution (obtained by superposition principal):

$$y = Ae^{(1+i)t} + Be^{(1-i)t}$$

- Using Euler's formula

$$\begin{aligned}y &= Ae^t \cos(t) + iAe^t \sin(t) + Be^t \cos(t) - iBe^t \sin(t) \\ &= (A + B)e^t \cos(t) + i(A - B)e^t \sin(t)\end{aligned}$$

- Here A and B should also be allowed to be *complex* and must be chosen to make y real

Our First Example

- Set $C = A + B$ and $D = i(A - B)$
- Then C and D are arbitrary constants and the general solution is

$$y = Ce^t \cos(t) + De^t \sin(t)$$

- **Note:** C and D must be real to make y real...

General Case

- Consider the equation

$$y'' + ay' + by = 0$$

- The corresponding characteristic polynomial will be $r^2 + ar + b$
- Roots come in conjugate pairs
- Suppose $r^2 + ar + b$ has the root $\alpha + \beta i$
- Then it will also have the root $\alpha - \beta i$
- Therefore the general (complex-valued) solution will look like

$$y = Ae^{(\alpha+i\beta)t} + Be^{(\alpha-i\beta)t}$$

General Case

- Euler's formula will then give

$$y = (A + B)e^{\alpha t} \cos(\beta t) + (A - B)e^{\alpha t} \sin(\beta t)i$$

- Setting $C = A + B$ and $D = (A - B)i$, the (real) general solution is

$$y = Ce^{\alpha t} \cos(\beta t) + De^{\alpha t} \sin(\beta t)$$

- Again C and D are completely arbitrary, but must be real-valued for y to be real.

Try it Yourself!

Find the general solutions:



$$y'' + 2y' - 8y = 0$$



$$y'' + 6y' + 13y = 0$$



$$y'' + 2y' + 1.25y = 0$$

Review!

Today:

- Complex numbers and Euler's definition
- What happens when the characteristic polynomial has complex roots

Next time:

- What happens when the characteristic polynomial has repeated roots