MATH 307: Problem Set #5

Due on: November 10, 2014

Problem 1 *Method of Undetermined Coefficients: General Solutions*

In each of the following, find the general solution of the given differential equation

(a) $y'' - 2y' - 3y = 3e^{2t}$

(b) $y'' - 2y' - 3y = -3te^{-t}$

(c) $y'' - 2y' - 3y = te^{-t} + 7e^{2t}$

(d) $y'' - 2y' - 3y = 2te^{-t} - 3e^{2t}$

(e) $y'' - 2y' - 3y = 4te^{-t} + e^{2t}$

(f) $y'' + 2y' + 5y = \sin(2t)$

(g) $y'' + 2y' + 5y = \cos(2t)$

(h) $y'' + 2y' + 5y = 4\sin(2t) + 7\cos(2t)$

(i) $y'' + 2y' = 3 + 4\sin(2t)$

(j) $y'' + 2y' + y = 2e^{-t}$

(k) $y'' + y = 3\sin(2t)$

(l) $y'' + y = t\cos(2t)$

(m) $y'' + y = 3\sin(2t) + t\cos(2t)$

(n) $y'' - y' - 2y = e^{t}$

(o) $y'' - y' - 2y = e^{-t}$

(p) $y'' - y' - 2y = \cosh(t)$ [Hint: $\cosh(t) = (e^{t} + e^{-t})/2$]

. . . .

Solution 1.

(a) The general solution of the corresponding homogeneous equation is

$$y_h = C_1 e^{3t} + C_2 e^{-t}.$$

Therefore, we must try a particullar solution of the form $y_p = Ae^{2t}$. Putting this into the differential equation, we find that $A = -1$. From this, we see that the general solution of the equation is

$$y = y_h + y_p = C_1 e^{3t} + C_2 e^{-t} - e^{2t}.$$

(b) The general solution of the corresponding homogeneous equation is

$$y_h = C_1 e^{3t} + C_2 e^{-t}.$$

Therefore, we must try a particullar solution of the form $y_p = (At^2 + Bt)e^{-t}$. Putting this into the differential equation, we find that $A = 3/8$ and $B = 3/16$. From this, we see that the general solution of the equation is

$$y = y_h + y_p = C_1 e^{3t} + C_2 e^{-t} + \left(\frac{3}{8}t^2 + \frac{3}{16}t\right) e^{-t}.$$

(c) Let y_1 be the particular solution found in part (a) and y_2 be the particular solution found in part (b), and let L be the linear differential operator

$$L\{y\} = y'' - 2y' - 3y.$$

Then the equation that we are trying to solve is

$$L\{y\} = te^{-t} + 7e^{2t}$$

Since L is a linear operator and $L\{y_1\} = 3e^{2t}$ and $L\{y_2\} = -3te^{-t}$, for any constants A and B we have that

$$L\{Ay_1 + By_2\} = AL\{y_1\} + BL\{y_2\} = 3Ae^{2t} + -3Bte^{-t}.$$

Thus if we choose $A = 7/3$ and $B = -1/3$, then we get a particular solution for our differential equation

$$y_p = \frac{7}{3}y_1 - \frac{1}{3}y_2 = -\frac{7}{3}e^{2t} + \left(\frac{-1}{8}t^2 + \frac{-1}{16}t\right) e^{-t}$$

Thus the general solution is

$$y = C_1 e^{3t} + C_2 e^{-t} - \frac{7}{3}e^{2t} + \left(\frac{-1}{8}t^2 + \frac{-1}{16}t\right) e^{-t}.$$

(d) Using a similar argument as in part (c), we get the general solution

$$y = C_1 e^{3t} + C_2 e^{-t} + e^{2t} + \left(\frac{-2}{8}t^2 + \frac{-2}{16}t\right) e^{-t}.$$

(e) Using a similar argument as in part (c), we get the general solution

$$y = C_1 e^{3t} + C_2 e^{-t} - \frac{1}{3}e^{2t} + \left(\frac{-4}{8}t^2 + \frac{-4}{16}t\right) e^{-t}.$$

(f) The general solution of the corresponding homogeneous equation is

$$y_h = C_1 e^{-t}\cos(2t) + C_2 e^{-t}\sin(2t)$$

To get a particular solution, we instead find a particular solution to the complex equation

$$\widetilde{y}'' + 2\widetilde{y}' + 5\widetilde{y} = e^{2it}.$$

To solve this equation, we try a solution of the form $\widetilde{y}_p = Ae^{2it}$. Putting this into the differential equationn, we find that

$$A = \frac{1}{1+4i} = \frac{1-4i}{17} = \frac{1}{17} - \frac{4}{17}i,$$

and therefore

$$\begin{aligned}\widetilde{y}_p &= \left(\frac{1}{17} - \frac{4}{17}i\right) e^{2t} \\ &= \left(\frac{1}{17} - \frac{4}{17}i\right)(\cos(2t) + i\sin(2t)) \\ &= \frac{1}{17}\cos(2t) + \frac{4}{17}\sin(2t) - \frac{4}{17}i\cos(2t) + \frac{1}{17}i\sin(2t)\end{aligned}$$

This means that

$$y_p = \operatorname{Im}\{\widetilde{y}_p\} = -\frac{4}{17}\cos(2t) + \frac{1}{17}\sin(2t)$$

Thus the general solution is

$$y = y_h + y_p = C_1 e^{-t}\cos(2t) + C_2 e^{-t}\sin(2t) - \frac{4}{17}\cos(2t) + \frac{1}{17}\sin(2t)$$

(g) The general solution to the corresponding homogeneous equation is the same as in (f). To get a particular solution, we instead find a particular solution to the complex equation

$$\widetilde{y}'' + 2\widetilde{y}' + 5\widetilde{y} = e^{2it}.$$

we solved this in part (f) and found

$$\widetilde{y}_p = \frac{1}{17}\cos(2t) + \frac{4}{17}\sin(2t) - \frac{4}{17}i\cos(2t) + \frac{1}{17}i\sin(2t)$$

Therefore

$$y_p = \operatorname{Re}\{\widetilde{y}_p\} = \frac{1}{17}\cos(2t) + \frac{4}{17}\sin(2t)$$

so that the general solution is

$$y = y_h + y_p = C_1 e^{-t}\cos(2t) + C_2 e^{-t}\sin(2t) + \frac{1}{17}\cos(2t) + \frac{4}{17}\sin(2t)$$

(h) The general solution to the corresponding homogeneous equation is the same as in (f) and (g). The particular solution of the equation will be a linear combination of the particular solutions found in (f) and (g). In particular, if y_1 is the particular solution found in (f) and y_2 is the particular solution found in (g), then the particular solution to the differential equation in (h) will be

$$y_p = 4y_1 + 7y_2 = \frac{-9}{17}\cos(2t) + \frac{32}{17}\sin(2t)$$

Thus the general solution is

$$y = y_h + y_p = C_1e^{-t}\cos(2t) + C_2e^{-t}\sin(2t) + \frac{-9}{17}\cos(2t) + \frac{32}{17}\sin(2t).$$

(i) The general solution to the corresponding homogeneous equation is

$$y_h = C_1 + C_2e^{-2t}.$$

To find a particular solution, we will find a particular solution of the equation

$$y_1'' + 2y_1' = 3$$

and a particular solution of the equation

$$y_2'' + 2y_2' = 4\sin(2t)$$

and then add them together: $y_p = y_1 + y_2$. To find y_1, notice that 3 is a solution of the homogeneous equation, so we should try $y_1 = At$. Doing so, we find $A = 3/2$, so that $y_1 = 3t/2$. Then to find y_2, we solve the corresponding complexified equation, like we did in problem (f). Doing so, we find $y_2 = -\frac{1}{2}\sin(2t) - \frac{1}{2}\cos(2t)$. Therefore the general solution is

$$y = y_h + y_p = y_h + y_1 + y_2 = C_1 + C_2e^{-2t} + \frac{3}{2}t - \frac{1}{2}\sin(2t) - \frac{1}{2}\cos(2t).$$

(j) The general solution to the corresponding homogeneous equation is

$$y_h = C_1e^{-t} + C_2te^{-t}.$$

Since e^{-t} corresponds to a double root of the characteristic polynomial of this equation, to get a particular solution we should try $y_p(t) = At^2e^{-t}$. Doing so, we find $A = 1$, and therefore the general solution is

$$y_h = C_1e^{-t} + C_2te^{-t} + t^2e^{-t}.$$

(k) The general solution of the corresponding homogeneous equation is

$$y_h = C_1\cos(t) + C_2\sin(t).$$

By complexifying and taking the imaginary component, we also find a particular solution of the form $y_p = -\sin(2t)$. Therefore the general solution is

$$y = y_h + y_p = C_1\cos(t) + C_2\sin(t) - \sin(2t).$$

(l) The general solution of the corresponding homogeneous equation is

$$y_h = C_1 \cos(t) + C_2 \sin(t).$$

To find a particular solution, we instead find a particular solution of the equation

$$\widetilde{y}'' + \widetilde{y} = te^{2it}$$

and take the real component. Doing so, we find a particular solution of the form $y_p = \frac{4}{9}\sin(2t) - \frac{1}{3}t\cos(2t)$. Therefore the general solution is

$$y = y_h + y_p = C_1 \cos(t) + C_2 \sin(t) + \frac{4}{9}\sin(2t) - \frac{1}{3}t\cos(2t).$$

(m) Let y_1 be the particular solution to part k and y_2 be the particular solution to part (l), then the particular solution here will be $y_p = y_1 + y_2$, so that

$$y_p = -\frac{5}{9}\sin(2t) - \frac{1}{3}t\cos(2t)$$

The general solution is therefore

$$y = y_h + y_p = C_1 \cos(t) + C_2 \sin(t) - \frac{5}{9}\sin(2t) - \frac{1}{3}t\cos(2t).$$

(n) The general solution of the corresponding homogeneous equation is

$$y_h = C_1 e^{2t} + C_2 e^{-t}.$$

To find a particular solution, we try $y_p = Ae^t$. Putting this into the differential equation, we find $A = -1/2$, so that the general solution is

$$y = y_h + y_p = C_1 e^{2t} + C_2 e^{-t} - \frac{1}{2}e^t.$$

(o) The general solution to the homogeneous equation is the same as in part (n). Since -1 is a root of the characteristic polynomial, to find the particular solution we try a solution of the form $y_p = Ate^{-t}$. Putting this into the differential equation, we find $A = 1/3$, so that the general solution is

$$y = y_h + y_p = C_1 e^{2t} + C_2 e^{-t} - \frac{1}{3}te^{-t}.$$

(p) The general solution to the homogeneous equation is the same as in part (n). If y_1 is the particular solution to part (n) and y_2 is the particular solution to part (o), then the particular solution here will be $\frac{1}{2}y_1 + \frac{1}{2}y_2$. Therefore the general solution is

$$y = y_h + y_p = C_1 e^{2t} + C_2 e^{-t} - \frac{1}{4}e^t - \frac{1}{6}te^{-t}.$$

Problem 2 *Method of Undetermined Coefficients: Initial Value Problems*

In each of the following, find the solution of the given initial value problem

(a) $y'' + 4y = t^2 + 3e^t$, $y(0) = 0$, $y'(0) = 2$

(b) $y'' - 2y' - 3y = 3te^{2t}$, $y(0) = 1$, $y'(0) = 0$

(c) $y'' + 2y' + 5y = 4e^{-t}\cos(2t)$, $y(0) = 1$, $y'(0) = 0$

. . . .

Solution 2.

(a) $y(t) = \frac{7}{10}\sin(2t) - \frac{19}{40}\cos(2t) - \frac{1}{8} + \frac{1}{4}t^2 + \frac{3}{5}e^t$

(b) $y(t) = e^{3t} + \frac{2}{3}e^{-t} + \left(\frac{-2}{3} - t\right)e^{2t}$

(c) $y(t) = e^{-t}\cos(2t) + \left(t + \frac{1}{2}\right)e^{-t}\sin(2t)$

Problem 3 *Another Problem...*

Determine the general solution of

$$y'' + \lambda^2 y = \sum_{m=1}^{N} a_m \sin(m\pi t),$$

where $\lambda > 0$ and $\lambda \neq m\pi$ for $m = 1, \ldots, N$.

. . . .

Solution 3. First of all, the general solution of the corresponding homogeneous equation

$$y'' + \lambda^2 y = 0$$

is $y = C_1\cos(\lambda t) + C_2\sin(\lambda t)$. Let y_m be a particular solution to the equation

$$y_m'' + \lambda^2 y_m = \sin(m\pi t)$$

Then $y_m = \operatorname{Im}\{\widetilde{y_m}\}$, where $\widetilde{y_m}$ is a particular solution of the complexified equation

$$\widetilde{y_m}'' + \lambda^2\widetilde{y_m} = e^{im\pi t}.$$

To solve this equation, we propose a solution of the form $\widetilde{y_m} = Ae^{im\pi t}$. Putting this into the differential equation, we find that $A = \frac{1}{\lambda^2 - m^2\pi^2}$, and therefore $\widetilde{y_m} = e^{im\pi t}/(\lambda^2 - m^2\pi^2)$. It follows that

$$y_m = \operatorname{Im}\{\widetilde{y_m}\} = \frac{1}{\lambda^2 - m^2\pi^2}\sin(m\pi t).$$

Using linearity properties, the particular solution to the original differential equation is therefore

$$y_m = \sum_{m=1}^{N} a_m y_m$$

so that the general solution is

$$y = C_1 \cos(\lambda t) + C_2 \sin(\lambda t) + \sum_{m=1}^{N} \frac{a_m}{\lambda^2 - m^2\pi^2} \sin(m\pi t).$$

Problem 4 *Differential Equations as Operators*

In this problem we indicate an alternative procedure for solving the differential equation

$$y'' + by' + cy = (D^2 + bD + c)y = g(t), \tag{1}$$

where b and c are constants, and D denotes differentiation with respect to t. Let r_1 and r_2 be the zeros of the characteristic polynomial of the corresponding homogeneous equation. These roots may be real and different, real and equal, or conjugate complex numbers.

(a) Verify that Eq (1) can be written in the factored form

$$(D - r_1)(D - r_2)y = g(t),$$

where $r_1 + r_2 = -b$ and $r_1 r_2 = c$.

(b) Let $u = (D - r_2)y$. Then show that the solution of Eq (1) can be found by solving the first order equations:

$$(D - r_1)u = g(t), \qquad (D - r_2)y = u(t).$$

. . . .

Solution 4.

(a) We calculate

$$(D - r_1)(D - r_2) = D^2 - r_1 D - r_2 D + r_1 r_2 = D^2 - (r_1 + r_2)D + r_1 r_2 = D^2 + bD + c.$$

(b) Suppose u is a solution of the equation $(D - r_1)u = g(t)$, and that y is a solution of the equation $(D - r_2)y = u(t)$. Then

$$y'' + by' + cy = (D^2 + bD + c)y = (D - r_1)(D - r_2)y = (D - r_1)u = g(t).$$

Hence y is a solution of Eqn. (1).

Problem 5 *Using the Previous Method...*

Using the method outlined in the previous problem, find the general solution to the following differential equations

(a) $y'' - 3y' - 4y = 3e^{2t}$

(b) $y'' + 2y' + y = 2e^{-t}$

....

Solution 5.

(a) We write $D^2 - 3D - 4$ as $(D-4)(D+1)$. We then solve the equation

$$(D-4)u = 3e^{2t}.$$

This equation is $u' - 4u = 3e^{2t}$, which has the integrating factor $\mu = e^{-4t}$. Using this, we find that the solution is

$$u = -\frac{3}{2}e^{2t} + C_1 e^{4t}.$$

We next solve the equation

$$(D+1)y = u.$$

This equation is $y' + y = -\frac{3}{2}e^{2t} + C_1 e^{4t}$, which has the integrating factor $\mu = e^t$. Using this, we find that the solution is

$$y = \frac{-1}{2}e^{2t} + \frac{C_1}{5}e^{4t} + C_2 e^{-t}.$$

Since C_1 and C_2 are arbitrary constants, we could instead write

$$y = \frac{-1}{2}e^{2t} + C_1 e^{4t} + C_2 e^{-t}.$$

(b) We write $D^2 + 2D + 1 = (D+1)^2$. We then solve the equation

$$(D+1)u = 2e^{-t}.$$

This equation is $u' + u = 2e^{-t}$, which has an integrating factor $\mu = e^t$. Using this, we find that the solution is

$$u = 2te^{-t} + C_1 e^{-t}.$$

We next solve the equation

$$(D+1)y = u.$$

This equation is $y' + y = 2te^{-t} + C_1 e^{-t}$, which has an integrating factor $\mu = e^t$. Using this, the solution is

$$y = t^2 e^{-t} + C_1 te^{-t} + C_2 e^{-t}.$$