# Math 307 Lecture 16

## Mechanical, Electrical, and Forced Vibrations

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November 12, 2014

# Today!

Last time:

- Damped and undamped vibrating springs

This time:

- More on Mechanical and Electrical Vibrations

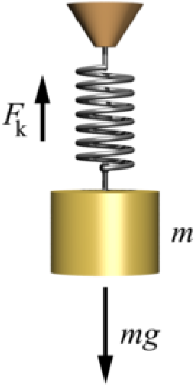
Next time:

- More on Forced Vibrations

# Outline

# Review of Damped Mass-Spring Systems

Figure : A Mass Spring System




- Equation of motion of a damped mass-spring system:

$$mu'' + \gamma u' + ku = 0.$$

- $u$: length of the mass spring system (relative to the resting length)
- $m$: mass (not weight!)
- $\gamma$: drag coefficient
- $k$: spring constant
- $m, \gamma, k$ all nonnegative

# Affect of Drag

- The value of $\gamma$ controls what the motion of the spring looks like!
- When $\gamma = 0$, we have an ideal system
- When $\gamma$ is small, we have a (weakly) damped mass spring system
- When $\gamma$ is large, we have an overdamped mass spring system
- We illustrate each of these motions with the next example!

# Example: Introduction

**Example**

Suppose a mass spring system has mass $m$ kg, spring constant $k$ N/m, and drag coefficient $\gamma$ N·s/m. The system is stretched 1 meter from its resting length and then released. Determine $u$ (its length relative to its resting length) as a function of time.

- $u$ (in meters) will be a solution to the initial value problem

$$mu'' + \gamma u' + ku = 0, \;\; u(0) = 1, \; u'(0) = 0.$$

- How does the solution to this IVP depend on $\gamma$?

# Example: When $\gamma = 0$

- Suppose $\gamma = 0$.
- Then the mass-spring system is *ideal*
- Satisfies the equation

$$mu'' + ku = 0, \;\; u(0) = 1, \; u'(0) = 0.$$

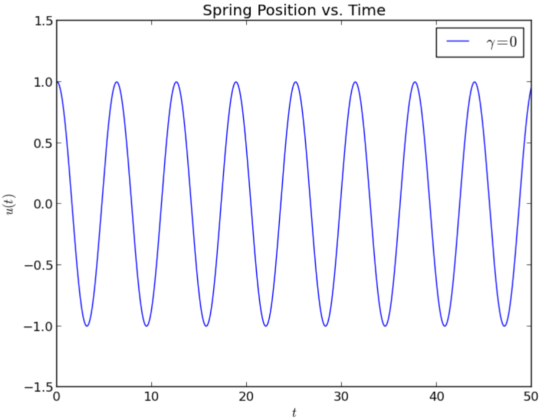
- Solution is (for $\omega = \sqrt{k/m}$)

$$u = \cos(\omega t)$$

- Check this!

# Plot of spring motion

Figure : Spring motion with no drag ($m = k = 1$, $\gamma = 0$)



# Example: When $\gamma$ is small

- Suppose $\gamma$ is small compared to $k$ and $m$
- To be concrete, let's take $m = k = 1$ and $\gamma = 0.2$
- Then the mass-spring system is *damped*
- Satisfies the equation

$$u'' + 0.2u' + u = 0, \;\; u(0) = 1, \;\; u'(0) = 0.$$

- Roots of the corresponding characteristic equation are

$$r = -0.1 \pm \sqrt{0.99}i$$

# Example: When $\gamma$ is small

- General solution is therefore

$$u = Ae^{-0.1t}\cos(\sqrt{0.99}t) + Be^{-0.1t}\sin(\sqrt{0.99}t)$$

- Initial conditions imply $A = 1$ and $B = 0.1/\sqrt{0.99}$ (check!)
- So the solution to the IVP is

$$u = e^{-0.1t}\cos(\sqrt{0.99}t) + \frac{0.1}{\sqrt{0.99}}e^{-0.1t}\sin(\sqrt{0.99}t)$$

- This form of the equation is harder to understand...
- We use some trig to write a simpler expression for this!

# Example: When $\gamma$ is small

- Remember: we can rewrite

$$A\cos(\mu t) + B\sin(\mu t)$$

- in the form

$$R\cos(\mu t - \delta)$$

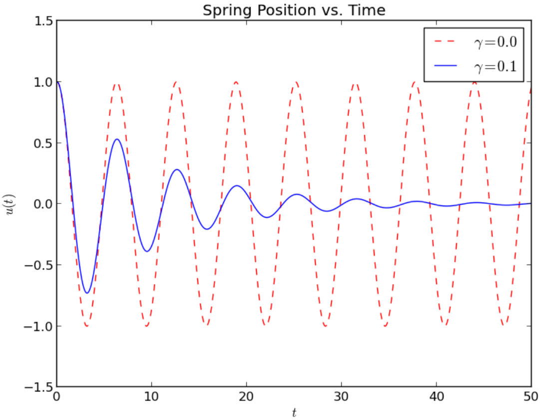
- by taking $R = \sqrt{A^2 + B^2}$, $\delta = \tan^{-1}(B/A)$.
- Using this, our previous solution is

$$u = e^{0.1t}\sqrt{\frac{100}{99}}\cos(\sqrt{0.99}t - \delta),$$

- for $\delta = \tan^{-1}(0.1/\sqrt{0.99}) \approx 0.10017$

# Plot of spring motion

Figure : Spring motion with no drag ($m = k = 1$, $\gamma = 0.1$)



# Example: When $\gamma$ is large

- Suppose $\gamma$ is large compared to $k$ and $m$
- To be concrete, let's take $m = k = 1$ and $\gamma = 2.5$
- Then the mass-spring system is *overdamped*
- Satisfies the equation

$$u'' + 2.5u' + u = 0, \;\; u(0) = 1, \; u'(0) = 0.$$

- Roots of the corresponding characteristic equation are

$$r_1 = -1/2, \;\; r_2 = -2$$

# Example: When $\gamma$ is large

- General solution is therefore

$$u = Ae^{-t/2} + Be^{-2t}$$

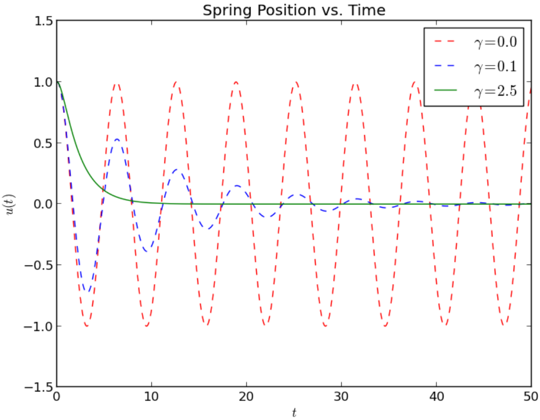
- Initial conditions imply $A = 4/3$ and $B = -1/3$ (check!)
- So the solution to the IVP is

$$u = \frac{4}{3}e^{-t/2} - \frac{1}{3}e^{-2t}$$

- This isn't trigonometric at all!
- That's why we call it overdamped

# Plot of spring motion

Figure : Spring motion with no drag ($m = k = 1$, $\gamma = 2.5$)



# How big is gamma for overdamping?

- Given a damped spring equation

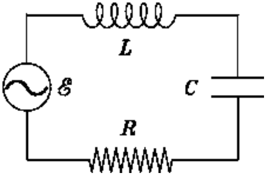
$$mu'' + \gamma u' + ku = 0,$$

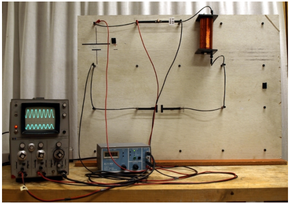
- The roots of the corresponding characteristic polynomial are

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}$$

- We get trig functions if and only if the discriminant $\gamma^2 - 4km$ is negative
- Therefore overdamping occurs when $\gamma \geq 2\sqrt{km}$

# LCR-circuits



- An LCR circuit involves a resistor, capacitor, inductor, and voltage source
- $L$: inductance of inductor (in henrys [H])
- $C$: capacitance of capacitor (in farads [F])
- $R$: resistance of resistor (in ohms [Ω])
- $E(t)$: voltage gain from energy source (in volts [V])

# LCR-circuits

- By Kirchhoff's law, the sum of the voltage drops and gains must be zero:

$$V_{\text{ind}} + V_{\text{cap}} + V_{\text{res}} + V_{\text{source}} = 0$$

- By convention, voltage drops are positive, and gains are negative ($V_{\text{source}} = -E(t)$)
- From elementary electromagnetism:
  - $V_{\text{ind}} = L\frac{dI}{dt}$
  - $V_{\text{cap}} = Q/C$
  - $V_{\text{res}} = IR$
- where $Q$ is the charge on the capacitor
- and $I = dQ/dt$ is the current in the circuit
- both $I$ and $Q$ are functions of time

# LCR-circuits

- Thus
$$L\frac{dI}{dt} + IR + Q/C - E(t) = 0.$$
- Differentiating with respect to time, and replacing $dQ/dt$ with $I$, this becomes
$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I - E'(t) = 0.$$
- In particular, when $E$ is constant, this equation becomes
$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = 0.$$
- Very similar to the equation of a damped spring system!

# A note on initial conditions

- To find the current as a function of time, we'll need to change this into an initial value problem
- Therefore, we'll want $I(0)$ (the initial current)
- and also $I'(0)$ (the initial derivative of current)
- Sometimes, we won't know $I'(0)$, but we will know the initial charge $Q$ on the capacitor
- Then we can use the equation

$$L\frac{dI}{dt} + RI + Q/C - E(t) = 0$$

to solve for $dI/dt$ at the initial time

# LCR-circuit example

## Example

Suppose that an LCR-circuit has a 1 H inductor, a 1 F capacitor, and a 0.125Ω resistor. Also suppose that the initial current on this circuit is 2 amp (A), and that the initial charge on the capacitor was 0.125 coulombs (C). If the source voltage $E = 0.25$ is constant, determine the current $I$ of the circuit as a function of time.

- We have $L = 1$, $C = 1$, and $R = 0.125$. Also $I(0) = 2$ and
$$I'(0) = \frac{E(0) - RI(0) - Q(0)/C}{L} = \frac{0.25 - 0.125 - 0.125}{1} = 0.$$
- So $I$ is a solution of the initial value problem
$$I'' + 0.125I' + I = 0, \quad I(0) = 2, \quad I'(0) = 0.$$

# LCR-circuit example

- We've solved this initial value problem before (on Monday), in the context of spring equations
- The roots of the characteristic polynomial are $\frac{-1}{16} \pm \frac{\sqrt{255}}{16} i$.
- This means that the general solution is

$$u = e^{-t/16}\left[A\cos\left(\frac{\sqrt{255}}{16}t\right) + B\sin\left(\frac{\sqrt{255}}{16}t\right)\right].$$

- To satisfy the initial conditions, we must choose $A = 2$ and $B = 2/\sqrt{255}$
- Then the solution of the initial value problem is

$$u = e^{-t/16}\left[2\cos\left(\frac{\sqrt{255}}{16}t\right) + \frac{2}{\sqrt{255}}\sin\left(\frac{\sqrt{255}}{16}t\right)\right].$$

# An Example

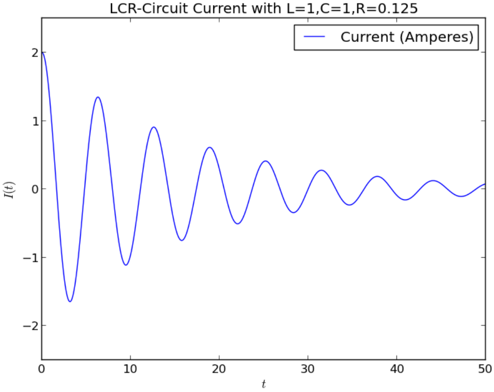
- Using our trig stuff with $R = \sqrt{A^2 + B^2}$ and $\delta = \tan^{-1}(B/A)$, we find

$$u = \frac{32}{\sqrt{255}} e^{-t/16} \cos\left(\frac{\sqrt{255}}{16} t - \delta\right)$$

- where $\delta = \tan^{-1} \frac{1}{\sqrt{255}} \approx 0.0625$

# Plot of spring motion

Figure : Current in the LCR circuit $L = 1$, $C = 1$, $R = 0.125$, $E = 0.25$, with the initial condition $E(0) = 2$ and $Q(0) = 0.125$



# Try it Yourself

## Example

An LCR-circuit has a capacitor of $0.25 \times 10^{-6}$ F and an inductor of 1 H, no resistor, and no source voltage. If the initial charge on the capacitor is $10^{-6}$ C, and there is no initial current, find the charge $Q$ on the capacitor at any time $t$.

- Give it a shot!
- Hint: it might be helpful to work with the equation

$$L\frac{dI}{dt} + RI + Q/C - E(t) = 0,$$

by replacing $I$ with $dQ/dt$

# Review!

Today:

- More on mechanical and electrical vibrations

Next time:

- Forced vibrations