Math 307 Lecture 18

More on Laplace Transforms

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Today!

Last time:

- More on Forced vibrations

This time:

- Intro to Laplace Transforms

Next time:

- Laplace Transforms and Step Functions

Outline

Basic Definition and Facts

Definition

Suppose $f(t)$ is some function. Then the function

$$\mathcal{L}\{f(t)\} := F(s) = \int_0^\infty e^{-st} f(t)dt$$

is called the *Laplace transform* of $f(t)$.

- The Laplace transform is a powerful tool for solving linear differential equations
- Before we find out how it is useful for this, let's look at some examples

Some Examples of Laplace Transforms

- Let $a > 0$, then

$$\begin{aligned}\mathcal{L}\left\{e^{at}\right\} &= \int_0^\infty e^{-st}e^{at}dt \\ &= \int_0^\infty e^{(a-s)t}dt \\ &= \frac{1}{a-s}e^{(a-s)t}\Big|_0^\infty \\ &= \frac{1}{s-a}\end{aligned}$$

Some Examples of Laplace Transforms

- Let $a \neq 0$, then

$$\begin{aligned}
\mathcal{L}\{\cos(at)\} &= \mathcal{L}\left\{\operatorname{Re}\left\{e^{iat}\right\}\right\} \\
&= \operatorname{Re}\left\{\mathcal{L}\left\{e^{iat}\right\}\right\} \\
&= \operatorname{Re}\left\{\int_0^\infty e^{-st}e^{iat}\,dt\right\} \\
&= \operatorname{Re}\left\{\frac{1}{s-ia}\right\} = \operatorname{Re}\left\{\frac{s+ia}{s^2+a^2}\right\} \\
&= \frac{s}{s^2+a^2}
\end{aligned}$$

Some Examples of Laplace Transforms

- Let $a \neq 0$, then

$$\begin{aligned}
\mathcal{L}\{\sin(at)\} &= \mathcal{L}\left\{\operatorname{Im}\left\{e^{iat}\right\}\right\} \\
&= \operatorname{Im}\left\{\mathcal{L}\left\{e^{iat}\right\}\right\} \\
&= \dots \\
&= \operatorname{Im}\left\{\frac{s+ia}{s^2+a^2}\right\} \\
&= \frac{a}{s^2+a^2}
\end{aligned}$$

Try it Yourself!

Find the Laplace Transform of the following functions...

- $f(t) = e^{at}\sin(bt)$
- $f(t) = e^{at}\cos(bt)$
- $f(t) = te^{at}$
- $f(t) = t\sin(at)$

Laplace Transform is a Linear Operator

- The Laplace transform is a “linear operator"
- In other words, given functions $f_1(t)$, $f_2(t)$ and constants c_1, c_2

$$\begin{aligned}\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} &= \int_0^\infty e^{-st}(c_1 f_1(t) + c_2 f_2(t))dt \\ &= c_1 \int_0^\infty e^{-st} f_1(t)dt + c_2 \int_0^\infty e^{-st} f_2(t)dt \\ &= c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}\end{aligned}$$

- In summary:

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}.$$

Laplace Transform Makes Derivatives into Polynomials

- The Laplace Transform changes expressions with derivatives in t to polynomials in s.
- For example

$$\begin{aligned}\mathcal{L}\left\{f'(t)\right\} &= \int_0^\infty e^{-st} f'(t)dt \\ &= e^{-st} f(t)\Big|_0^\infty - \int_0^\infty (-s)e^{-st} f(t)dt \\ &= -f(0) + s\int_0^\infty e^{-st} f(t)dt = s\mathcal{L}\left\{f(t)\right\} - f(0)\end{aligned}$$

- In summary:

$$\mathcal{L}\left\{f'(t)\right\} = s\mathcal{L}\left\{f\right\} - f(0).$$

Laplace Transform Makes Derivatives into Polynomials

- How about $\mathcal{L}\left\{f''(t)\right\}$?
- Well, from the previous identity applied to $f'(t)$:

$$\mathcal{L}\left\{f''(t)\right\} = s\mathcal{L}\left\{f'(t)\right\} - f'(0)$$

- Then since $\mathcal{L}\left\{f'(t)\right\} = s\mathcal{L}\left\{f\right\} - f(0)$, we find

$$\mathcal{L}\left\{f''(t)\right\} = s(s\mathcal{L}\left\{f\right\} - f(0)) - f'(0) = s^2\mathcal{L}\left\{f\right\} - sf(0) - f'(0).$$

Try it Yourself!

Rewrite each of the following in terms of s and $\mathcal{L}\{f\}$

- $\mathcal{L}\left\{f^{(4)}(t)\right\}$
- $\mathcal{L}\{f'' + 2f' + 4f\}$

Laplace Transform Makes Derivatives into Polynomials

More generally, we have the following identity (check!)

Equation

$$\mathcal{L}\left\{f^{(n)}\right\} = s^n\mathcal{L}\{f\} - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

A Word of Caution to this Tale...

- Not every function has a Laplace transform!
- For example, consider the function $f(t) = e^{t^2}$
- Notice that $e^{-st}e^{t^2} = e^{t^2-st} \to +\infty$ for large t, regardless of the value of s
- Therefore the integral

$$\mathcal{L}\left\{e^{t^2}\right\} = \int_0^\infty e^{-st}e^{t^2}\,dt \;\; \text{DOES NOT EXIST}$$

- This means that e^{t^2} does not have a Laplace transform.
- Can you think of any other functions without a Laplace transform?

When Should We Expect Laplace Transforms to Exist?

- To describe what kinds of functions have Laplace transforms, we need a couple definitions:

Definition

A function $f(t)$ is *of exponential type* if there exists positive constants K, a such that $|f(t)| \leq Ke^{at}$

Definition

A function $f(t)$ is *piecewise continuous* if on any finite interval $[a, b]$ it is the multipart rule of a finite number of continuous functions bounded on $[a, b]$

When Should We Expect Laplace Transforms to Exist?

- With this we have the following theorem

Theorem

Suppose $f(t)$ is piecewise continuous of exponential type, and that $K, a > 0$ satisfy $|f(t)| \leq Ke^{at}$. Then $\mathcal{L}\{f(t)\} = F(s)$ exists for all $s > a$.

- If you are trying to take the Laplace transform of a function that is not piecewise continuous or of exponential type, then you should be suspicious!

Inverse of a Laplace Transform

- One of the most important properties of Laplace transforms, is that they are invertible

Definition

Consider any function $F(s)$, and suppose that there exists a piecewise continuous function of exponential type $f(t)$ such that $\mathcal{L}\{f(t)\} = F(s)$. Then $f(t)$ is unique, and is called the *inverse Laplace transform* of $F(s)$, and denoted $\mathcal{L}^{-1}\{F(s)\}$

Inverse Example 1

Example

Find the inverse Laplace transform of $F(s) = \frac{2}{s^2+4}$

- Recall that $\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}$
- This means that $\mathcal{L}\{\sin(2t)\} = \frac{2}{s^2+4}$
- Therefore $\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \sin(2t)$

Inverse Example 2

Example

Find the inverse Laplace transform of $F(s) = \frac{s}{s^2+2s+6}$

- Hmm...this form doesn't look familiar. Any ideas?
- Best idea: complete the square!
- $s^2 + 2s + 6 = (s+1)^2 + 4$
- Where does this get us?
- Remember that

$$\mathcal{L}\left\{e^{at}\sin(bt)\right\} = \frac{b}{(s-a)^2+b^2}, \quad \mathcal{L}\left\{e^{at}\cos(bt)\right\} = \frac{(s-a)}{(s-a)^2+b^2}$$

Inverse Example 2

- We want to put $F(s)$ in this form.
- Doing so is easy!

$$
\begin{aligned}
F(s) &= \frac{s}{s^2+2s+6} = \frac{s}{(s+1)^2+4} \\
&= \frac{(s+1)-1}{(s+1)^2+4} \\
&= \frac{(s+1)}{(s+1)^2+4} - \frac{1}{(s+1)^2+4} \\
&= \frac{(s+1)}{(s+1)^2+4} - \frac{1}{2}\frac{2}{(s+1)^2+4}
\end{aligned}
$$

Inverse Example 2

- Now since $\mathcal{L}\left\{e^t\cos(2t)\right\} = \frac{(s+1)}{(s+1)^2+4}$ and $\mathcal{L}\left\{e^t\sin(2t)\right\} = \frac{2}{(s+1)^2+4}$
- We have shown that (since $\mathcal{L}\left\{\cdot\right\}$ is linear)

$$\begin{aligned} F(s) &= \mathcal{L}\left\{e^t\cos(2t)\right\} - \frac{1}{2}\mathcal{L}\left\{e^t\sin(2t)\right\} \\ &= \mathcal{L}\left\{e^t\cos(2t) - \frac{1}{2}e^t\sin(2t)\right\} \end{aligned}$$

- Therefore

$$\mathcal{L}^{-1}\left\{F(s)\right\} = e^t\cos(2t) - \frac{1}{2}e^t\sin(2t).$$

Inverse Example 3

Example

Find the inverse Laplace transform of $F(s) = \frac{s}{s^2+2s+1}$

- What should we try?
- This time, factor!
- $s^2 + 2s + 1 = (s+1)^2$
- Then use partial fractions

Inverse Example 3

- First we write

$$F(s) = \frac{s}{(s+1)^2} = \frac{A}{s+1} + \frac{B}{(s+1)^2}$$

- This tells us $s = A(s+1) + B$
- When $s = -1$, this gives us $B = -1$
- Comparing coefficients of s on both sides also tells us $A = 1$
- Therefore

$$F(s) = \frac{1}{s+1} - \frac{1}{(s+1)^2}$$

Inverse Example 3

- For n a positive integer, the Laplace transform of $t^n e^{at}$ is (by a homework problem)

$$\mathcal{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}}$$

- Therefore $\mathcal{L}\left\{e^{-t}\right\} = \frac{1}{s+1}$ and $\mathcal{L}\left\{te^{-t}\right\} = \frac{1}{(s+1)^2}$
- This means

$$F(s) = \mathcal{L}\left\{e^{-t}\right\} - \mathcal{L}\left\{te^{-t}\right\} = \mathcal{L}\left\{e^{-t} - te^{-t}\right\}$$

- and therefore

$$\mathcal{L}^{-1}\left\{F(s)\right\} = e^{-t} - te^{-t}$$

Solving IVPs with Laplace Transforms

- Suppose we have an IVP

$$ay'' + by' + cy = f(t).$$

- We can solve this using the Laplace transform
- The idea is to take the Laplace transform of both sides
- Then find an expression for $\mathcal{L}\{y\}$
- Then invert it to find y

Solving IVPs Example 1

Example

Solve the IVP

$$y'' - y' - 2y = 0, \;\; y(0) = 1, \; y'(0) = 0$$

using Laplace transforms.

- First of all

$$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0) = s\mathcal{L}\{y\} - 1.$$

- and also

$$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0) = s^2\mathcal{L}\{y\} - s.$$

Solving IVPs Example 1

- Taking the Laplace transform of both sides of the differential equation yields

$$\mathcal{L}\{y'' - y' - 2y\} = \mathcal{L}\{0\} = 0$$

- Moreover

$$\begin{aligned}\mathcal{L}\{y'' - y' - 2y\} &= \mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} \\ &= (s^2 - s - 2)\mathcal{L}\{y\} + 1 - s\end{aligned}$$

- Therefore

$$(s^2 - s - 2)\mathcal{L}\{y\} + 1 - s = 0,$$

so that

$$\mathcal{L}\{y\} = \frac{s-1}{s^2 - s - 2}$$

Solving IVPs Example 1

- Factoring, we get

$$s^2 - s - 2 = (s-2)(s+1)$$

- Then partial fractions tells us

$$\mathcal{L}\{y\} = \frac{s-1}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$$

where

$$s - 1 = A(s+1) + B(s-2)$$

- When $s = -1$, this tells us that $B = 2/3$, and when $s = 2$, this tells us that $A = 1/3$ Therefore

$$\mathcal{L}\{y\} = \frac{1/3}{s-2} + \frac{2/3}{s+1}$$

Solving IVPs Example 1

- Thus

$$\mathcal{L}\{y\} = \frac{1}{3}\mathcal{L}\left\{e^{2t}\right\} + \frac{2}{3}\mathcal{L}\left\{e^{-t}\right\}$$
$$= \mathcal{L}\left\{\frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}\right\}$$

and therefore

$$y = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}.$$

Try it Yourself!

Use the Laplace transform to solve the following initial value problems!

- $y'' + y = \sin(2t),\ \ y(0) = 2,\ y'(0) = 1$
- $y^{(4)} - y = 0,\ \ y(0) = 0,\ y'(0) = 1,\ y''(0) = 0,\ y'''(0) = 0$

Review!

Today:

- Fun with Laplace Transforms!

Next time:

- Laplace Transforms and Step Functions