Math 307 Lecture 19

Laplace Transforms of Discontinuous Functions

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Today!

Last time:

- Step Functions

This time:

- Laplace Transforms of Discontinuous Functions
- Differential Equations with Discontinuous Forcing

Next time:

- More on Discontinuous Forcing

Outline

Step Functions and Hat Functions

- Last time we defined a step function to be a function of the form

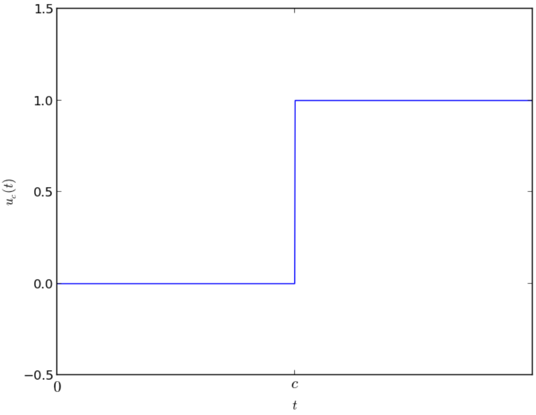
$$u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases}$$

- And a hat function to be a function of the form

$$h_{a,b}(t) = u_a(t) - u_b(t) = \begin{cases} 0, & \text{if } t < a \\ 1, & \text{if } a \leq t < b \\ 0, & \text{if } t > b \end{cases}$$

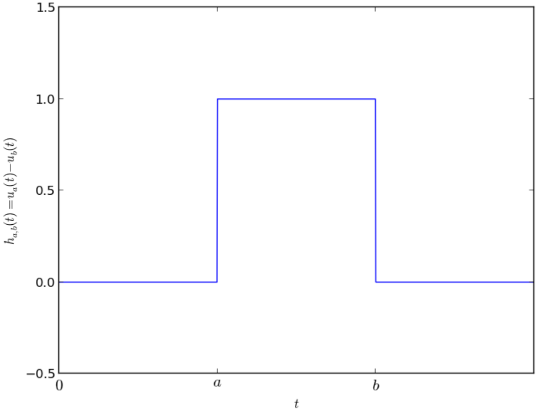
Plot of a Step Function

Figure : Plot of step function $u_c(t)$



Plot of a Hat Function

Figure : Plot of hat function $h_{a,b}(t)$



Brackets to Step Functions Example

- Last time we also learned how to convert from a function defined this way:

$$f(t) = \begin{cases} \sin(t) & \text{if } 0 \leq t < \pi/4 \\ \sin(t) + \cos(t - \pi/4) & \text{if } t \geq \pi/4 \end{cases}$$

- To a function defined this way

$$f(t) = \sin(t)u_0(t) + \cos(t - \pi/4)u_{\pi/4}(t)$$

Try it Yourself!

Convert the following functions from bracket form to step function form:

- $$f(t) = \begin{cases} 1 & \text{if } 0 \leq t < 2 \\ e^{-(t-2)} & \text{if } t \geq 2 \end{cases}$$

- $$f(t) = \begin{cases} t & \text{if } 0 \leq t < 1 \\ t-1 & \text{if } 1 \leq t < 2 \\ t-2 & \text{if } 2 \leq t < 3 \\ 0 & \text{if } t \geq 3 \end{cases}$$

Laplace Transform of $f(t-c)u_c(t)$

- Suppose that $f(t)$ is a piecewise continuous functions of exponential type, and that $c > 0$. Then
- We wish to calculate the Laplace transform of $f(t-c)u_c(t)$
- Computation first shows

$$\mathcal{L}\{f(t-c)u_c(t)\} = \int_0^\infty e^{-st} f(t-c)u_c(t)dt$$
$$= \int_c^\infty e^{-st} f(t-c)dt$$

Laplace Transform of $f(t-c)u_c(t)$

- Now if we do the u-substitution $u = t - c$, then $du = dt$ and

$$\mathcal{L}\{f(t-c)u_c(t)\} = \int_0^\infty e^{-s(u+c)} f(u)du$$
$$= e^{-sc}\int_0^\infty e^{-su} f(u)du = e^{-sc}\mathcal{L}\{f(t)\}.$$

- To summarize

$$\mathcal{L}\{f(t-c)u_c(t)\} = e^{-sc}\mathcal{L}\{f(t)\}.$$

- And consequently for $\mathcal{L}\{f(t)\} = F(s)$,

$$\mathcal{L}^{-1}\left\{F(s)e^{-sc}\right\} = f(t-c)u_c(t)$$

Laplace Transform of $e^{ct}f(t)$

- Let $F(s)$ be the Laplace transform of $f(t)$
- We wish to calculate the Laplace transform of $f(t)e^{ct}$
- Computation first shows

$$\mathcal{L}\left\{e^{ct}f(t)\right\} = \int_0^\infty e^{-st}e^{ct}f(t)dt$$
$$= \int_0^\infty e^{-(s-c)t}f(t)dt = F(s-c)$$

- To summarize $\mathcal{L}\left\{f(t)e^{ct}\right\} = F(s-c)$,
- And consequently $\mathcal{L}^{-1}\left\{F(s-c)\right\} = e^{ct}f(t)$

Summary of Additional Laplace Transform Properties

Additional Laplace transform properties:

- $$\mathcal{L}\left\{f(t-c)u_c(t)\right\} = e^{-sc}\mathcal{L}\left\{f(t)\right\}$$
- $$\mathcal{L}^{-1}\left\{e^{-sc}F(s)\right\} = f(t-c)u_c(t)$$
- $$\mathcal{L}\left\{f(t)e^{ct}\right\} = F(s-c)$$
- $$\mathcal{L}^{-1}\left\{F(s-c)\right\} = e^{ct}f(t)$$

A First Example

Example

Find the Laplace transform of

$$f(t) = \begin{cases} \sin(t) & \text{if } 0 \leq t < \pi/4 \\ \sin(t) + \cos(t - \pi/4) & \text{if } t \geq \pi/4 \end{cases}$$

- As we saw earlier, we may write

$$f(t) = \sin(t)u_0(t) + \cos(t - \pi/4)u_{\pi/4}(t)$$

A First Example

- So what will it's Laplace transform look like?
- Using the Laplace transform of step functions, we see that its Laplace transform is

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \mathcal{L}\{\sin(t)u_0(t)\} + \mathcal{L}\left\{\cos(t-\pi/4)u_{\pi/4}(t)\right\} \\ &= e^{-0s}\mathcal{L}\{\sin(t)\} + e^{-\pi s/4}\mathcal{L}\{\cos(t)\} \\ &= \frac{1}{s^2+1} + e^{-\pi s/4}\frac{s}{s^2+1}\end{aligned}$$

A Second Example

Example

Find the inverse Laplace transform of

$$F(s) = \frac{1 - e^{-2s}}{s^2}$$

- We first write

$$F(s) = \frac{1}{s^2} - e^{-2s}\frac{1}{s^2}$$

- Also recall that $\mathcal{L}\{t\} = \frac{1}{s^2}$

A Second Example

- Therefore we calculate

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{s^2}\right\}$$
$$= t - u_2(t)(t-2)$$

- We can but this back into the bracket notation as

$$f(t) = \begin{cases} t & \text{if } t < 2 \\ 2 & \text{if } t \geq 2 \end{cases}$$

A Third Example

Example

Find the inverse Laplace transform of

$$F(s) = \frac{1}{s^2 - 4s + 5}$$

- We first write

$$F(s) = \frac{1}{(s-2)^2 + 1}$$

- Therefore we can write $F(s) = G(s-2)$ for $G(s) = \frac{1}{s^2+1}$
- Since $\mathcal{L}^{-1}\{G(s)\} = \sin(t)$, it follows that

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{G(s-2)\} = e^{2t}\sin(t).$$

Laplace Transform Examples

Find the Laplace Transforms of the following functions

- $$f(t) = \begin{cases} 0 & \text{if } t < 2 \\ (t-2)^2 & \text{if } t \geq 2 \end{cases}$$

- $$f(t) = \begin{cases} 0 & \text{if } t < \pi \\ t - \pi & \text{if } \pi \leq t < 2\pi \\ 0 & \text{if } t \geq 2\pi \end{cases}$$

Inverse Laplace Transform Examples

Find the Inverse Laplace Transforms of the following functions

- $$F(s) = \frac{3!}{(s-2)^4}$$
- $$F(s) = \frac{2(s-1)e^{-2s}}{s^2-2s+2}$$
- $$F(s) = \frac{(s-2)e^{-s}}{s^2-4s+3}$$

An Example

Example

Find the solution of the initial value problem

$$2y'' + y' + 2y = g(t),\ \ y(0) = 0,\ \ y'(0) = 0$$

where

$$g(t) = u_5(t) - u_{20}(t) = \begin{cases} 0 & \text{if } 0 \leq t < 5 \\ 1 & \text{if } 5 \leq t < 20 \\ 0 & \text{if } t \geq 20 \end{cases}$$

An Example

- This IVP models the charge on a capacitor in an LCR circuit where a battery is connected at $t = 5$ and disconnected at $t = 20$
- We calculate using $y(0) = 0$ and $y'(0) = 0$

$$\mathcal{L}\left\{y'\right\} = s\mathcal{L}\left\{y\right\} - y(0) = s\mathcal{L}\left\{y\right\}.$$

- and

$$\mathcal{L}\left\{y''\right\} = s^2\mathcal{L}\left\{y\right\} - sy(0) - y'(0) = s^2\mathcal{L}\left\{y\right\}.$$

An Example

- Moreover

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{u_5(t)\} - \mathcal{L}\{u_20\}(t) = e^{-5s}\frac{1}{s} - e^{-20s}\frac{1}{s}$$

- so taking the Laplace transform of both sides of the original differential equation

$$2y'' + y' + 2y = g(t)$$

- gives us

$$2s^2\mathcal{L}\{y\} + s\mathcal{L}\{y\} + 2\mathcal{L}\{y\} = \frac{e^{-5s} - e^{-20s}}{s}.$$

An Example

- After a little algebra, this tells us

$$\mathcal{L}\{y\} = \frac{e^{-5s} - e^{-20s}}{s(2s^2+s+2)} = (e^{-5s} - e^{-20s})H(s)$$

- for $H(s) = 1/(s(2s^2+s+2))$
- Thus if $h(t) = \mathcal{L}^{-1}\{H(s)\}$, then

$$\begin{aligned} y &= \mathcal{L}^{-1}\left\{(e^{-5s} - e^{-20s})H(s)\right\} \\ &= \mathcal{L}^{-1}\{e^{-5s}H(s)\} - \mathcal{L}^{-1}\left\{e^{-20s}H(s)\right\} \\ &= h(t-5)u_5(t) - h(t-20)u_{20}(t) \end{aligned}$$

An Example

- Now using partial fractions

$$\frac{1}{s(2s^2+s+2)} = \frac{a}{s} + \frac{bs+c}{2s^2+s+2},$$

- one easily determines that $a = 1/2, b = 1$ and $c = 1/2$, so

$$H(s) = \frac{1/2}{s} - \frac{s+\frac{1}{2}}{2s^2+s+2}$$

An Example

- The inverse Laplace transform of $\frac{1/2}{s}$ is easy
- The inverse Laplace transform of $\frac{s+\frac{1}{2}}{2s^2+s+2}$ is more difficult
- How do we find it?
- Complete the square in the denominator

$$\frac{s+\frac{1}{2}}{2s^2+s+2} = \frac{s+\frac{1}{2}}{2(s+\frac{1}{4})^2+\frac{15}{8}}$$

- Factor out a $1/2$

$$\frac{s+\frac{1}{2}}{2s^2+s+2} = \frac{1}{2}\frac{s+\frac{1}{2}}{(s+\frac{1}{4})^2+\frac{15}{16}}$$

An Example

- Lastly, try to put this in a form that looks like a linear combination of translations of Laplace transforms of $\sin(\sqrt{15}t/4)$ and $\cos(\sqrt{15}t/4)$

$$\begin{aligned}\frac{s+\frac{1}{2}}{2s^2+s+2} &= \frac{1}{2}\frac{(s+\frac{1}{4})+\frac{1}{4}}{(s+\frac{1}{4})^2+\frac{15}{16}} \\ &= \frac{1}{2}\frac{(s+\frac{1}{4})}{(s+\frac{1}{4})^2+\frac{15}{16}} + \frac{1}{2}\frac{\frac{1}{4}}{(s+\frac{1}{4})^2+\frac{15}{16}} \\ &= \frac{1}{2}\frac{(s+\frac{1}{4})}{(s+\frac{1}{4})^2+\frac{15}{16}} + \frac{1}{2\sqrt{15}}\frac{\frac{\sqrt{15}}{4}}{(s+\frac{1}{4})^2+\frac{15}{16}}\end{aligned}$$

An Example

- Therefore

$$\frac{s+\frac{1}{2}}{2s^2+s+2} = \frac{1}{2}\mathcal{L}\left\{e^{-t/4}\cos(\sqrt{15}t/4)\right\} + \frac{1}{2\sqrt{15}}\mathcal{L}\left\{e^{-t/4}\sin(\sqrt{15}t/4)\right\}$$

An Example

- Thus we have shown that

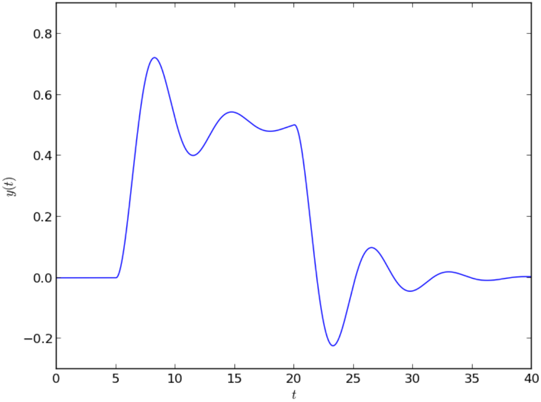
$$y = h(t-5)u_5(t) - h(t-20)u_{20}(t)$$

- for

$$h(t) = \frac{1}{2} - \frac{1}{2}e^{-t/4}\cos(\sqrt{15}t/4) - \frac{1}{2\sqrt{15}}e^{-t/4}\sin(\sqrt{15}t/4)$$

Plot of a Solution to IVP

Figure : Plot of Solution to IVP with Discontinuous Forcing



Review!

Today:

- Fun with Laplace Transforms!

Next time:

- More fun with Laplace Transforms!