

# Math 307 Lecture 1

## Introducing Differential Equations!

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Department of Mathematics  
University of Washington

March 30, 2013

# Today!

## Plan for today:

- What is a differential equation?
- First order differential equations
- Separable and homogeneous equations

## Next time:

- Linear equations
- Method: Integrating factors
- Method: Variation of parameters

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# What's a Differential Equation?

## Question

What is a differential equation?

## Definition

*A differential equation* is mathematical expression describing a relationship between a function and its derivatives

Before going further we should think about:

- Examples of differential equations
- Why we care about differential equations
- The scope of our study of diff. equations in MATH 307

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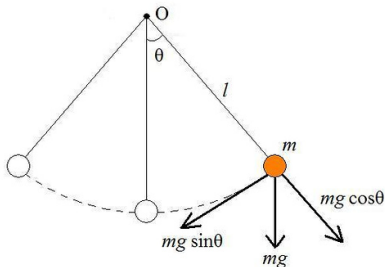


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# Example Diff. Eqn: Motion of a Rigid Pendulum

Figure : A physics-type picture you've probably seen before



- Newton's second law:  
 $\tau = I \frac{d^2\theta}{dt^2}$
- Torque:  $\tau = mgl \sin \theta$
- Moment of inertia:  
 $I = ml^2$
- We get a differential equation!

$$\frac{d^2\theta}{dt^2} = \frac{mg}{l} \sin \theta$$

## Example Diff. Eqn: Compound interest

**Figure :** A traditional celebration of compound interest as demonstrated by the notable entrepreneur Scrooge Mc. Duck



- For continuously compounded interest

$$\frac{dS}{dt} = rS$$

- $S$  is invested capital
- $r$  is interest rate
- This is a differential equation!
- Solution is  $S(t) = S_0 e^{rt}$   
(How do we get this?)

## Example Diff. Eqn: Falling with air drag

**Figure :** Diving to the earth from the stratosphere is probably more fun than differential equations. Maybe?



- Newton's second law:  
 $F = ma$
- Using a linear drag model

$$m \frac{d^2 y}{dt^2} = -mg + k \frac{dy}{dt}$$

- $y$  is your height
- $g$  is gravitational acceleration
- $k$  is a drag coefficient
- How can we solve this equation to get  $y$ ?

## Example Diff. Eqn: Fluid flow in one dimension

**Figure :** A fluid flow is as cool as it is complicated! Below is an example of what are called Von Karman vortices. Caveat: this flow is two-dimensional



- Goal: find velocity of the fluid  $u = u(x, t)$
- $x$  is position in the fluid
- $t$  is time
- $p$  is pressure
- $\rho$  is density

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{1}{\rho} \frac{d^2 p}{dx^2}$$

- It's a *partial differential equation* because it has partial derivatives

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# The What and Why of MATH 307

**Q:** Why should we learn about differential equations?

**A:** As we have already seen, they naturally come up all over the place!

**Q:** So this course will make me a super-max-boss-pro-stud at everything to do with differential equations?

**A:** Unfortunately, it is impossible to learn everything in one quarter. Instead, we will focus on what are called first and second order equations.

**Q:** Ah, so at least I will be able to dominate these so-called first and second order equations?

**A:** Again no. Differential equations, even first and second order ones, can be really hard to solve! Our goal: learn to identify ones which are easy to solve and how

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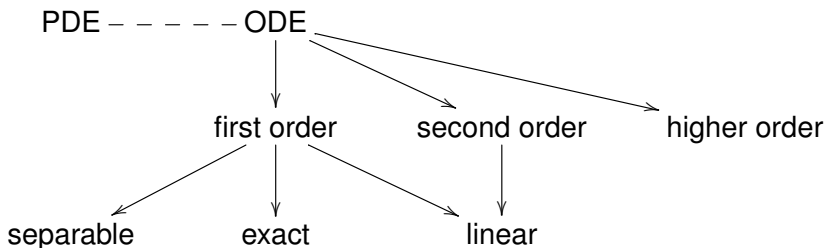
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# Classification of Differential Equations

Figure : A MATH 307 perspective of the "types" of differential equations



# First Order Equations

## Definition

A first order ordinary differential equation is an equation of the form

$$\frac{dy}{dt} = f(t, y)$$

where  $f$  is a function of the two variables  $t$  and  $y$ .

- Our goal is to find *solutions* to first order differential equations
- Algebraically, this means find a function  $y(t)$  satisfying the above equation
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# Slope Fields

- Consider the first order equation

$$\frac{dy}{dt} = f(t, y)$$

- Make a "grid" of points in the  $x, y$ -plane
- At each grid point  $(x, y)$ , draw a dash with slope  $f(x, y)$
- This slope field "represents" our differential equation
- Solutions to the equation are curves whose tangent slopes agree with the dashes!
- Solution curves are determined uniquely by a point they contain (sometimes called an initial condition)

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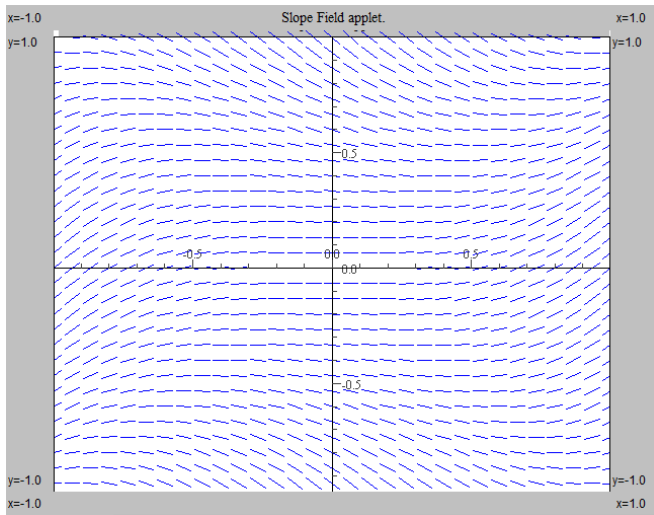
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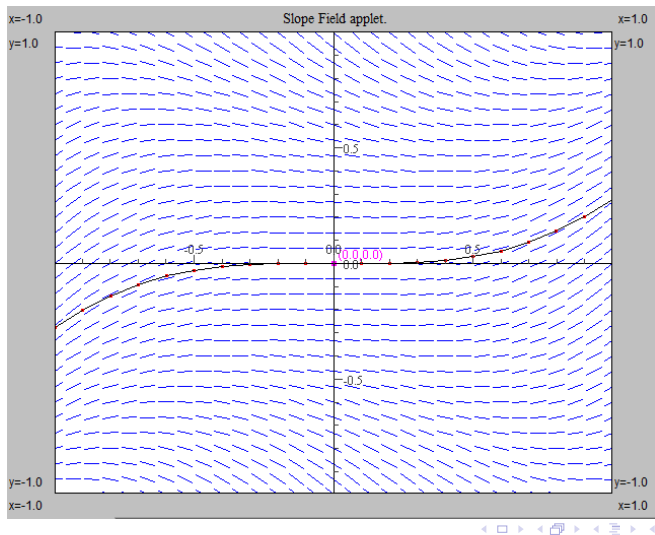
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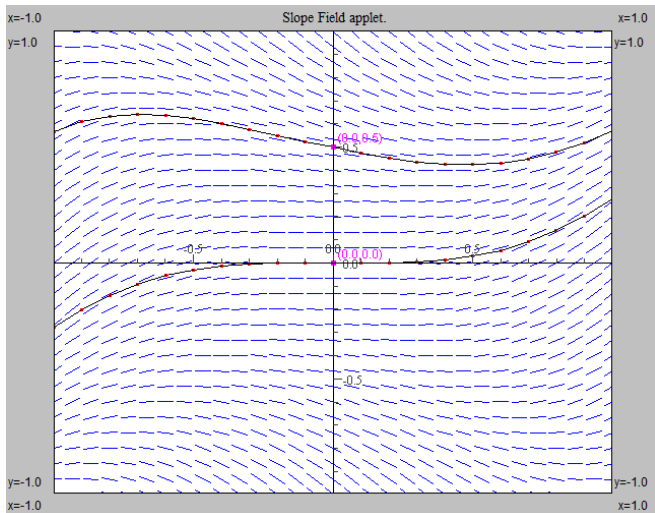
# Slope Field Example: $\frac{dy}{dt} = x^2 - y^2$



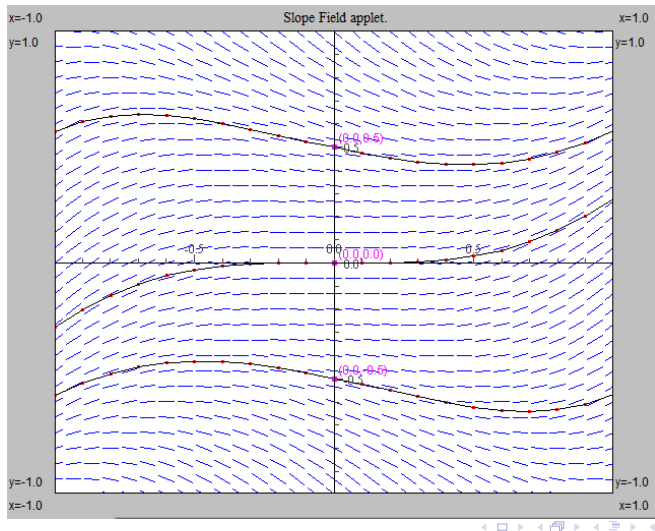
# Slope Field Example: Solution satisfying $y(0) = 0$



# Slope Field Example: Solution satisfying $y(0) = 0.5$



# Slope Field Example: Solution satisfying $y(0) = -0.5$



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# Separable Equation

## Definition

A first order differential equation

$$\frac{dy}{dx} = f(x, y)$$

is called *separable* if  $f(x, y) = g(x)h(y)$  for some functions  $g, h$

Examples:

- $\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}$

- $\frac{dy}{dx} = \frac{(y-3)\cos x}{1+2y^2}$

- $y' = (e^{-x} - e^x)/(3 + 4y)$

- $\sin(2x)dx + \cos(3y)dy = 0$

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## Question

How can we solve a separable equation?

$$\frac{dy}{dx} = g(x)h(y)$$

$$\frac{1}{h(y)} dy = g(x) dx$$

$$\int \frac{1}{h(y)} dy = \int g(x) dx$$

... finish by solving for y

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## Example

Find a solution to the differential equation  $y' = (1 - 2x)y^2$  satisfying the initial condition  $y(0) = -1/6$ .

$$\begin{aligned} \frac{1}{y^2}y' &= (1 - 2x) \\ \int \frac{1}{y^2}dy &= \int (1 - 2x)dx \\ -\frac{1}{y} &= x - x^2 + C \\ y &= \frac{-1}{x - x^2 + C} \end{aligned}$$

$y(0) = -1/6$  implies  $C = 6$ . Hence  $y = \frac{-1}{x - x^2 + 6}$ .

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## Definition

A homogeneous equation is a first order differential equation of the form

$$\frac{dy}{dx} = f(x, y),$$

where  $f(x, y) = g(y/x)$  for some function  $g$ .

- Homogeneous equations have "scale invariant" slope fields: if you zoom out, the slope field looks the same!
- Homogeneous equations are separable equations in disguise!

Examples:

- $\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}$

- $y' = \frac{3y^2 - x^2}{2xy}$

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# Solving Homogeneous Equations

Steps to solve:

- Do enough algebra to write  $\frac{dy}{dx} = g(y/x)$
- Define a new variable  $z = y/x$
- Since  $xz = y$ , implicit differentiation says

$$z + x \frac{dz}{dx} = \frac{dy}{dx}$$

- Plugging back into the original DE, we get a separable equation

$$z + x \frac{dz}{dx} = g(z)$$

- Solve this separable equation for  $z$  and use  $y = xz$  to WIN

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# An Example Worked Out

## Question

Find a solution to the differential equation  $y' = \frac{x^2 + xy + y^2}{x^2}$  satisfying the initial condition  $y(1) = 0$ .

- Notice that  $y' = 1 + \frac{y}{x} + \frac{y^2}{x^2} = g(y/x)$  for  $g(z) = 1 + z + z^2$
- If we set  $z = y/x$ , then we find  $z + x \frac{dz}{dx} = 1 + z + z^2$

$$\frac{dz}{dx} = \frac{1 + z^2}{x}$$

$$\frac{1}{1 + z^2} dz = \frac{1}{x} dx$$

$$\int \frac{1}{1 + z^2} dz = \int \frac{1}{x} dx$$



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- If we set  $z = y/x$ , then we find  $z + x \frac{dz}{dx} = 1 + z + z^2$

$$\frac{dz}{dx} = \frac{1 + z^2}{x}$$
$$\frac{1}{1 + z^2} dz = \frac{1}{x} dx$$
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## An Example Worked Out

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## An Example Worked Out ~ Continued

$$\arctan(z) = \ln|x| + C$$

$$z = \tan(\ln|x| + C)$$

$$y = xz = x \tan(\ln|x| + C)$$

- Since  $y(1) = 0$ , we must have  
 $0 = 1 \tan(\ln|1| + C) = \tan(C)$ .
- This tells us  $C = 0$ . Hence the solution we want is

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## What we did today:

- We learned what a differential equation is and why we should care
- We caught a glimpse of first order differential equations
- We learned how to solve separable and homogeneous equations

## Plan for next time:

- Linear equations
- Method: Integrating factors
- Method: Variation of parameters

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