Math 307 Lecture 2

First Order Linear Equations!

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Today!

Last time:

- What is a differential equation?
- First order differential equations
- Separable and homogeneous equations

This time:

- First-Order Linear Equations
- Exact Equations
- Solving Exact, Linear Equations

Next time:

- More First-Order Linear Equations
- Method: Integrating Factors
- Method: Variation of Parameters

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Outline

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1. First Order Linear Equations
   - What are they, and why should we care?
   - Our very first example!
2. Exact Equations
   - Exact Equations

What is a First-Order Linear Equation?

Recall that a first order ordinary differential equation (ODE) is an equation of the form

$$\frac{dy}{dt} = f(x, y).$$

Question

When is a first order differential equation *linear*?

Definition

A first order ODE is called *linear* if it can be written in the form

$$y' = p(x)y + q(x)$$

for some functions $p(x), q(x)$

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Examples of Linear Equations

- Linear: $y' = 2xy + 3\sin(x)$
- Not linear: $y' = y^2$
- Linear: $y' = \cos(3x)y + e^x$
- Not linear: $yy' = x^2$
- Linear: $xy' + x^2y = \sin(x)$ (divide by x on both sides)

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Why do we care about First-Order Linear Equations?

Why should we care about first-order linear ODE?

- Arises naturally in many situations
  - Newtons law of cooling
  - Compound interest
  - Mixing fluids in a tank
  - Velocity of a falling body with air friction
- We know how to solve them!
  - General solution for linear: involves one arbitrary constant
  - Not true for nonlinear equations (see for example $y' = y^2$)
- Can approximate solutions of nonlinear equations by solutions of linear ones
  - For example, consider the IVP $\frac{dy}{dt} = e^{ty}$; $y(0) = 0$
  - By Taylor series $e^{ty} \approx 1 + ty$
  - Solutions to IVP $\frac{dy}{dt} = 1 + ty$; $y(0) = 0$ are approximations

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Outline

A First Example

Example

Let a, b be *constants*. Find a general solution to the ODE

$$\frac{dy}{dt} = ay + b$$

- We already know how to solve this! Why?
- That's right, it's separable!

$$\frac{1}{ay+b}\frac{dy}{dt} = 1$$

$$\frac{1}{ay+b}dy = dt$$

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A First Example ∼ Continued

Continuing our calculations...

$$\int \frac{1}{ay+b}dy = \int dt$$

$$\frac{1}{a}\ln|ay+b| = t + C_0$$

$$\ln|ay+b| = at + C_1 \qquad C_1 = aC_0$$

$$ay+b = e^{at+C_1}$$

$$ay+b = C_2e^{at} \qquad C_2 = e^{C_1}$$

$$y = C_3e^{at} - \frac{b}{a} \qquad C_3 = C_2/a$$

We might in the future drop the indexing of the constants, and just let arbitrary constants be arbitrary :)

A First Example ∼ Continued

Continuing our calculations...

$$\begin{aligned}
\int \frac{1}{ay+b}dy &= \int dt \\
\frac{1}{a}\ln|ay+b| &= t + C_0 \\
\ln|ay+b| &= at + C_1 && C_1 = aC_0 \\
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Solving Arbitrary First-Order linear ODEs

- In general, solving first-order linear equations won't be as easy :(
- Even so, we will always be able to solve them (up to an integral)
- We will give two methods for this next lecture:
  - Method of Integrating Factors
  - Method of Variation of Parameters
- Their names come from more general methods
- You should be careful to know how to solve an equation both ways!
- TODAY: how to solve *exact* linear equations

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1. First Order Linear Equations
   - What are they, and why should we care?
   - Our very first example!

2. Exact Equations
   - Exact Equations

Motivating Example

Example

Find the general solution of the first-order linear ODE

$$\sin(x)y' + \cos(x)y = \sec(x)\tan(x)$$

- How might we solve this?
- Observe: $\sin(x)y' + \cos(x)y = (\sin(x)y)'$
- This means: $(\sin(x)y)' = \sec(x)\tan(x)$
- Integrating: $\sin(x)y = \sec(x) + C$

Final answer:

$$y = \frac{\sec(x) + C}{\sin(x)}$$

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Exact Equations

That was cool, right?

- One question: What just happened ?!?!
- Answer: the differential equation was *exact*

What's that suppose to mean?

- Given any first-order ODE $y' = f(x, y)$
- We can always rewrite it in the form

$$M(x, y) + N(x, y)y' = 0$$

Definition

A first order ODE $M(x, y) + N(x, y)y' = 0$ is exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

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Definition

A first order ODE $M(x, y) + N(x, y)y' = 0$ is exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Exact Equations

That was cool, right?

- One question: What just happened ?!?!
- Answer: the differential equation was *exact*

What's that suppose to mean?

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Exact Equation Practice!

Let's try to tell if the following equations are exact

- $y\cos(xy) + x\cos(xy)y' = 0$
- Answer: yes!
- $2xy^2 + x^2yy' = 0$
- Answer: no!
- $e^t y' = e^t y + \frac{1}{1+t^2}$
- Answer: yup!
- Solutions to exact equations are easy to find...
- Today, we'll only talk about how to solve exact linear equations

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Solving Exact Linear Equations

Consider a first order linear equation of the form

$$a(x)y' + b(x)y = c(x).$$

In M, N-form, this equation is

$$\overbrace{b(x)y - c(x)}^{M(x,y)} + \overbrace{a(x)}^{N(x,y)} y' = 0$$

- This equation is exact if $a'(x) = b(x)$ (why?)
- In this case

$$(a(x)y)' = a'(x)y + a(x)y' = b(x)y + a(x)y'$$

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Thus in the exact case, we can rewrite the original equation

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If we now integrate both sides with respect to x:

$$a(x)y = \int c(x)dx.$$

so the solution is

$$y = \frac{1}{a(x)} \int c(x)dx.$$

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Find the solution to the equation

$$e^x y' + e^x y = \cos(x).$$

- This equation is exact
- In particular, $(e^x y)' = e^x y' + e^x y$
- Original equation becomes: $(e^x y)' = \cos(x)$
- Integrating: $e^x y = \sin(x) + C$
- Solution: $y = e^{-x} \sin(x) + Ce^{-x}$

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- We learned about linear equations
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Plan for next time:

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