MATH 307: Problem Set #1

Due on: April 11, 2013

Problem 1 *First Order Linear Equations*

For each of the following equations:

(i) Use a computer to graph the slope field of the differential equation.
Include a printout of your graph with your homework

(ii) Based on inspection of the direction field, describe how you expect solutions to behave for large values of t

(iii) Find the general solution to the equation and use it to determine how the solution behaves as $t \to \infty$.

(a) $y' - 2y = t^2 e^{2t}$

(b) $ty' + 2y = \sin(t)$, for $t > 0$

(c) $ty' - y = t^2 e^{-t}$, for $t > 0$

. . . .

Solution 1.

(a) Suppose that $\mu = \mu(t)$ is our integrating factor. Then

$$\mu(t)y' - 2\mu(t)y = \mu(t)t^2 e^{2t}$$

is exact. Putting this in M, N-notation

$$\overbrace{-\mu(t)t^2 e^{2t} - 2\mu(t)y}^{M(t,y)} + \overbrace{\mu(t)}^{N(t,y)}\, y' = 0$$

exactness means that $\partial M/\partial y = \partial N/\partial t$, and therefore

$$-2\mu(t) = \mu'(t)$$

This differential equation is easy to solve! A solution is $\mu(t) = e^{-2t}$. Thus the equation

$$e^{-2t}y' - 2e^{-2t}y = t^2$$

is exact! We can group stuff together on the left hand side to then get

$$(e^{-2t}y)' = t^2.$$

Now integrating both sides with respect to t, we find

$$\begin{aligned}\int (e^{-2t}y)'dt &= \int t^2 dt\\ e^{-2t}y &= \frac{1}{3}t^3 + C\\ y &= \frac{1}{3}t^3e^{2t} + Ce^{2t}\end{aligned}$$

(b) Suppose that $\mu = \mu(t)$ is our integrating factor. Then

$$\mu(t)ty' + 2y\mu(t) = \mu(t)\sin(t)$$

is exact. Putting this into M, N notation, we then get

$$\overbrace{2\mu(t)y - 2\mu(t)\sin(t)}^{M(t,y)} + \overbrace{\mu(t)t}^{N(t,y)}\,y' = 0.$$

exactness means that $\partial M/\partial y = \partial N/\partial t$, and therefore

$$2\mu(t) = \mu(t) + t\mu'(t).$$

This simplifies to

$$\mu(t) = t\mu'(t).$$

This equation really easy! A solution is $\mu(t) = t$. Thus the equation

$$t^2y' + 2ty = t\sin(t)$$

is exact! We can group the things on the left together to find

$$(t^2y)' = t\sin(t)$$

Now integrating both sides with respect to t, we find

$$\begin{aligned}\int (t^2y)'dt &= \int t\sin(t)dt\\ t^2y &= \sin(t) - t\cos(t) + C\\ y &= t^{-2}\sin(t) - t^{-1}\cos(t) + Ct^{-2}\end{aligned}$$

(c) Suppose that $\mu = \mu(t)$ is our integrating factor. Then

$$\mu(t)ty' - \mu(t)y = \mu(t)t^2e^{-t}$$

is exact. Putting this in M, N-notation

$$\overbrace{-\mu(t)y - \mu(t)t^2e^{-t}}^{M(t,y)} + \overbrace{t\mu(t)}^{N(t,y)}\, y' = 0$$

exactness means that $\partial M/\partial y = \partial N/\partial t$, and therefore

$$-\mu(t) = \mu(t) + t\mu'(t)$$

which simplifies to

$$-2\mu(t) = t\mu'(t).$$

This differential equation is a quickie, and a solution is $\mu(t) = t^{-2}$. Thus the equation

$$t^{-1}y' - t^{-2}y = e^{-t}$$

is exact! We can group things together on the left hand side to get

$$(t^{-1}y)' = e^{-t}$$

Integrating both sides, we find

$$\begin{aligned} \int (t^{-1}y)'dt &= \int e^{-t}dt \\ t^{-1}y &= -e^{-t} + C \\ y &= -te^{-t} + Ct \end{aligned}$$

Problem 2 *First Order Linear Initial Value Problems*

Find the solution to each of the given initial value problems

(a) $y' + 2y = te^{-2t}$, $y(1) = 0$

(b) $y' + 2y/t = \frac{1}{t^2}\cos(t)$, $y(\pi) = 0$

(c) $ty' + 2y = \sin(t)$, $y(\pi/2) = 1$

(d) $t^3y' + 4t^2y = e^{-t}$, $y(-1) = 0$

. . . .

Solution 2.

(a) Multiply by integrating factor $\mu(t)$:

$$\mu(t)y' + 2\mu(t)y = \mu(t)te^{-2t}$$

Put into M, N-notation form:

$$\overbrace{2\mu(t)y - \mu(t)te^{-2t}}^{M(t,y)} + \overbrace{\mu(t)}^{N(t,y)} y' = 0$$

Exactness implies that $\partial M/\partial y = \partial N/\partial t$ and therefore

$$2\mu(t) = \mu'(t)$$

A solution of this equation is $\mu(t) = e^{2t}$. Thus we have

$$e^{2t}y' + 2e^{2t}y = t.$$

Gathering together the y-terms, we get

$$(e^{2t}y)' = t.$$

Lastly we integrate and solve for y:

$$\begin{aligned}\int (e^{2t}y)'dt &= \int tdt \\ e^{2t}y &= \frac{1}{2}t^2 + C \\ y &= \frac{1}{2}t^2e^{-2t} + Ce^{-2t}\end{aligned}$$

Since $y(1) = 0$, we get

$$0 = \frac{1}{2}1^2e^{-2(1)} + Ce^{-2(1)},$$

and therefore $C = -\frac{1}{2}$. Thus the solution is

$$y(t) = \frac{1}{2}t^2e^{-2t} - \frac{1}{2}e^{-2t}$$

(b) Multiply by integrating factor $\mu(t)$:

$$\mu(t)y' + 2\mu(t)y/t = \mu(t)\frac{1}{t^2}\cos(t)$$

In M, N notation, this is

$$\overbrace{2\mu(t)y/t - \mu(t)\frac{1}{t^2}\cos(t)}^{M(t,y)} + \overbrace{\mu(t)}^{N(t,y)} y' = 0$$

Exactness implies that $\partial M/\partial y = \partial N/\partial t$ and therefore

$$2\mu(t)/t = \mu'(t)$$

and a solution to this is $\mu(t) = t^2$. Thus we have the exact equation

$$t^2 y' + 2ty = \cos(t)$$

Exactness of this equation allows us to gether all the y parts to get

$$(t^2 y)' = \cos(t)$$

Integrating then leads to

$$\begin{aligned} \int (t^2 y)' dt &= \int \cos(t) dt \\ t^2 y &= \sin(t) + C \\ y &= t^{-2}\sin(t) + Ct^{-2} \end{aligned}$$

Since $y(\pi) = 0$, we also know that

$$0 = \pi^{-2}\sin(\pi) + C\pi^{-2}$$

and therefore $C = 0$. The solution is therefore

$$y = t^{-2}\sin(t).$$

(c) Multiply by integrating factor $\mu(t)$:

$$\mu(t)ty' + 2\mu(t)y = \mu(t)\sin(t)$$

In M, N notation form

$$\overbrace{2\mu(t)y - \mu(t)\sin(t)}^{M(t,y)} + \overbrace{\mu(t)t}^{N(t,y)}\, y' = 0$$

Exactness implies that $\partial M/\partial y = \partial N/\partial t$ and therefore

$$2\mu(t) = \mu(t) + t\mu'(t).$$

This simplifies to

$$\mu(t) = t\mu'(t).$$

A solution of this equation is $\mu(t) = t$ Thus we have the exact equation

$$t^2 y' + 2ty = t\sin(t)$$

Exactness gives us the power to gather together the y parts:

$$(t^2 y)' = t\sin(t).$$

Integrating both sides, we find

$$\begin{aligned}\int (t^2y)'dt &= \int t\sin(t)dt \\ t^2y &= \sin(t) - t\cos(t) + C \\ y &= t^{-2}\sin(t) - t^{-1}\cos(t) + Ct^{-2}\end{aligned}$$

Now using the fact that $y(\pi/2) = 1$ we find

$$1 = (\pi/2)^{-2}\sin(\pi/2) - (\pi/2)^{-1}\cos(\pi/2) + C(\pi/2)^{-2},$$

and therefore $C = \frac{\pi^2}{4} - 1$. Thus the solution is

$$y = t^{-2}\sin(t) - t^{-1}\cos(t) + \left(\frac{\pi^2}{4} - 1\right)t^{-2}$$

(d) Multiply by integrating factor $\mu(t)$:

$$\mu(t)t^3y' + 4\mu(t)t^2y = \mu(t)e^{-t}$$

In M, N-notation form this is

$$\overbrace{4\mu(t)t^2y - \mu(t)e^{-t}}^{M(t,y)} + \overbrace{\mu(t)t^3}^{N(t,y)}y' = 0$$

Exactness implies that $\partial M/\partial y = \partial N/\partial t$ and therefore

$$4\mu(t)t^2 = \mu'(t)t^3 + 3t^2\mu(t)$$

which simplifies to

$$\mu(t) = \mu'(t)t.$$

A solution to this equation is $\mu(t) = t$, and therefore we have the exact equation

$$t^4y' + 4t^3y = te^{-t}$$

Since this is exact, we can gather the y parts:

$$(t^4y)' = te^{-t}$$

Now integrating both sides with respect to t, we find

$$\begin{aligned}\int (t^4y)'dt &= \int te^{-t}dt \\ t^4y &= -te^{-t} - e^{-t} + C \\ y &= -t^{-3}e^{-t} - t^{-4}e^{-t} + Ct^{-4}\end{aligned}$$

Using the fact that $y(-1) = 0$, we then obtain

$$0 = -(-1)^{-3}e^1 - (-1)^{-4}e^1 + C(-1)^{-4}$$

and therefore $C = 0$. Hence the solution is

$$y = -t^{-3}e^{-t} - t^{-4}e^{-t}.$$

Problem 3 *Variation of Parameters*

Use variation of parameters to find the general solution of the given differential equation

(a) $y' + y/t = 3\cos(2t)$

(b) $2y' + y = 3t^2$

. . . .

Solution 3.

(a) We rewrite this in $y' = p(t)y + q(t)$ notation:

$$y' = \overbrace{-\frac{1}{t}}^{p(t)} y + \overbrace{3\cos(2t)}^{q(t)}.$$

Let y_h be a solution to the corresponding homogeneous equation. That is:

$$y_h' = -\frac{1}{t} y_h$$

A solution to this equation is $y_h(t) = \frac{1}{t}$. Then the method of variation of parameters tells us that $y = v(t)y_h(t)$, where

$$v(t) = \int \frac{q(t)}{y_h(t)} dt = \int 3t\cos(2t)dt = \frac{3}{2}t\sin(2t) + \frac{3}{4}\cos(2t) + C$$

Hence

$$y(t) = \frac{3}{2}\sin(2t) + \frac{3}{4}t^{-1}\cos(2t) + Ct^{-1}$$

(b) We rewrite this in $y' = p(t)y + q(t)$ notation:

$$y' = \overbrace{-\frac{1}{2}}^{p(t)} y + \overbrace{\frac{3}{2}t^2}^{q(t)}.$$

Let y_h be a solution to the corresponding homogeneous equation. That is:

$$y_h' = -\frac{1}{2} y_h$$

A solution to this equation is $y_h(t) = e^{-t/2}$. Then the method of variation of parameters tells us that $y = v(t)y_h(t)$, where

$$v(t) = \int \frac{q(t)}{y_h(t)} dt = \int \frac{3}{2}t^2 e^{t/2} dt = 3t^2 e^{t/2} - 12te^{t/2} + 24e^{t/2} + C$$

Hence

$$y(t) = 3t^2 - 12t + 24 + Ce^{-t/2}$$

Problem 4 *Separable Equations*

In each of the following, find a family of solutions parametrized by a constant

(a) $y' = \frac{x^2}{y(1+x^3)}$

(b) $y' + y^2 \sin(x) = 0$

(c) $\frac{dy}{dx} = \frac{x-e^{-x}}{y+e^y}$

(d) $\frac{dy}{dx} = \frac{x^2}{1+y^2}$

....

Solution 4.

(a) Using the usual steps of separating and integrating, we find

$$\begin{aligned} yy' &= \frac{x^2}{1+x^3} \\ ydy &= \frac{x^2}{1+x^3}dx \\ \int ydy &= \int \frac{x^2}{1+x^3}dx \\ \frac{1}{2}y^2 &= \frac{1}{3}\log|1+x^3| + C \\ y^2 &= \frac{2}{3}\log|1+x^3| + C \\ y &= \pm\sqrt{\frac{2}{3}\log|1+x^3| + C} \end{aligned}$$

(b) Using the usual steps of separating and integrating, we find

$$\begin{aligned} y' &= y^2\sin(x) \\ \frac{1}{y^2}y' &= \sin(x) \\ \frac{1}{y^2}dy &= \sin(x)dx \\ \int \frac{1}{y^2}dy &= \int \sin(x)dx \\ -\frac{1}{y} &= -\cos(x) + C \\ y &= \frac{1}{\cos(x)+C} \end{aligned}$$

(c) Using the usual steps of separating and integrating, we find

$$\begin{aligned}(y+e^y)\frac{dy}{dx} &= x - e^{-x}\\ (y+e^y)dy &= (x-e^{-x})dx\\ \int (y+e^y)dy &= \int (x-e^{-x})dx\\ \frac{1}{2}y^2 + e^y &= \frac{1}{2}x + e^{-x} + C\end{aligned}$$

This equation is too hard to solve for y, so we leave it in this form.

(d) Using the usual steps of separating and integrating, we find

$$\begin{aligned}(1+y^2)\frac{dy}{dx} &= x^2\\ (1+y^2)dy &= x^2dx\\ \int (1+y^2)dy &= \int x^2dx\\ y + \frac{1}{3}y^3 &= \frac{1}{3}x^3 + C\end{aligned}$$

This equation is too hard to solve for y, so we leave it in this form.

Problem 5 *Separable Initial Value Problems*

For each of the following initial value problems

(i) Solve the initial value problem

(ii) Using a computer, graph the solution
Attach a printout of your graph to your homework

(iii) Determine as accurately as you can the interval in which the solution is defined

(a) $xdx + ye^{-x}dy = 0$, $y(0) = 1$

(b) $y' = \frac{3x^2 - e^x}{2y-5}$, $y(0) = 1$

. . . .

Solution 5.

(a) We start out by finding a big family of solutions

$$\begin{aligned} x dx &= -y e^{-x} dy \\ x e^{x} dx &= -y dy \\ \int x e^{x} dx &= \int -y dy \\ x e^{x} - e^{x} + C &= -\frac{1}{2} y^{2} \\ y &= \pm\sqrt{-2x e^{x} + 2e^{x} + C} \end{aligned}$$

Now from the initial condition, we have $y(0) = 1$, so we know that we want the $+$ in front of the square root above. Also this means that

$$1 = \sqrt{-2(0)e^{0} + 2e^{0} + C},$$

and therefore $C = -1$. We conclude

$$y = \sqrt{-2x e^{x} + 2e^{x} - 1}$$

This solution is defined for $-2xe^{x} + 2e^{x} - 1 \geq 0$, or for x between approximately -1.678 and 0.768.

(b) We start out by finding a big family of solutions

$$\begin{aligned} (2y-5)y' &= 3x^{2} - e^{x} \\ (2y-5)dy &= (3x^{2} - e^{x})dx \\ \int (2y-5)dy &= \int (3x^{2} - e^{x})dx \\ y^{2} - 5y &= x^{3} - e^{x} + C \\ y &= \frac{5 \pm \sqrt{25 + 4(x^{3} - e^{x} + C)}}{2} \end{aligned}$$

Now $y(0) = 1$ tells us that we want the negative sign in front of the square root! Also it says

$$1 = \frac{5 - \sqrt{21 + 4C}}{2}$$

and therefore $C = -3$. Hence the solution is

$$y = \frac{5 - \sqrt{25 + 4x^{3} - 4e^{x} - 12}}{2}$$

This is defined for $25 + 4x^{3} - 4e^{x} - 12 \geq 0$ or x between approximately -1.4445 and 4.6297.

Problem 6 *Homogeneous Equations*

For each of the following, show that the equation is homogeneous. Then find a family of solutions differing by a constant

(a) $\frac{dy}{dx} = \frac{3y^2 - x^2}{2xy}$

(b) $\frac{dy}{dx} = -\frac{4x+3y}{2x+y}$

. . . .

Solution 6.

(a) Since

$$\frac{dy}{dx} = \frac{3}{2}(y/x) - \frac{1}{2}(x/y)$$

it is clear that this differential equation is homogeneous. By setting $z = y/x$, we know that $y = xz$, and therefore

$$\frac{dy}{dx} = z + x\frac{dz}{dx}.$$

Thus the old differential equation may be rewritten as

$$z + x\frac{dz}{dx} = \frac{3}{2}z - \frac{1}{2}\frac{1}{z}.$$

This is separable! We then solve for z in the usual way

$$\begin{aligned} x\frac{dz}{dx} &= \frac{1}{2}z - \frac{1}{2}\frac{1}{z} \\ x\frac{dz}{dx} &= \frac{1}{2}\frac{z^2-1}{z} \\ 2\frac{z}{z^2-1}\frac{dz}{dx} &= \frac{1}{x} \\ 2\frac{z}{z^2-1}dz &= \frac{1}{x}dx \\ \int 2\frac{z}{z^2-1}dz &= \int \frac{1}{x}dx \\ \ln|z^2-1| &= \ln|x| + C \\ z^2 - 1 &= Cx \\ z^2 &= Cx + 1 \\ z &= \pm\sqrt{Cx+1} \end{aligned}$$

Since $y = xz$, it then follows

$$y = \pm x\sqrt{Cx+1}.$$

(b) Since

$$\frac{dy}{dx} = -\frac{4 + 3y/x}{2 + y/x}$$

it is clear that this differential equation is homogeneous. By setting $z = y/x$, we know that $y = xz$ and therefore

$$\frac{dy}{dx} = z + x\frac{dz}{dx}.$$

Thus the old differential equation may be rewritten as

$$z + x\frac{dz}{dx} = -\frac{4 + 3z}{2 + z}$$

This is separable! We then solve for z in the usual way

$$\begin{aligned}
x\frac{dz}{dx} &= -\frac{4 + 3z}{2 + z} - z \\
x\frac{dz}{dx} &= -\frac{4 + 5z + z^2}{2 + z} \\
-\frac{2 + z}{4 + 5z + z^2}\frac{dz}{dx} &= \frac{1}{x} \\
-\left(\frac{1}{3}\frac{1}{z+1} + \frac{2}{3}\frac{1}{z+4}\right)\frac{dz}{dx} &= \frac{1}{x} \\
-\left(\frac{1}{3}\frac{1}{z+1} + \frac{2}{3}\frac{1}{z+4}\right)dz &= \frac{1}{x}dx \\
-\int\left(\frac{1}{3}\frac{1}{z+1} + \frac{2}{3}\frac{1}{z+4}\right)dz &= \int\frac{1}{x}dx \\
-\left(\frac{1}{3}\log|z+1| + \frac{2}{3}\log|z+4|\right) &= \log|x| + C \\
\log|z+1| + 2\log|z+4| &= -3\log|x| + C \\
\log|z+1| + \log(z+4)^2 &= -3\log|x| + C \\
\log|(z+1)(z+4)^2| &= -3\log|x| + C \\
(z+1)(z+4)^2 &= Cx^{-3}
\end{aligned}$$

This is as far as we can reasonably simplify. Now using the fact that $z = y/x$, we get the equation

$$\left(\frac{y}{x} + 1\right)\left(\frac{y}{x} + 4\right)^2 = Cx^{-3}$$

Multiplying both sides by x^3 makes this look a bit better

$$(y + x)(y + 4x)^2 = C.$$