Math 307 Quiz 3

May 18, 2014

Problem 1. Without solving the equation, find the largest interval on which we can immediately expect a unique solution to the initial value problem

$$y' = \frac{1}{1-x}y + \csc(x), \quad y(0.1) = 42.$$

Use a sentence or two to explain your reasoning.

Solution 1. The function $\csc(x)$ has singularities at integer multiples of π, ie. $x = \ldots, -2\pi, -\pi, 0, \pi, 2\pi$. The function $\frac{1}{1-x}$ has a singularity at $x = 1$. Therefore the largest open interval containing 0.1 where these two functions are well-behaved is $(0, 1)$. By our existence and uniqueness theorem for first-order linear equations, we therefore have a unique solution to this initial value problem on the interval $(0, 1)$.

Problem 2. Give an example of an initial value problem with no solution.

Solution 2.

$$y' = \frac{1}{x}, \quad y(0) = 1.$$

Problem 3. Give an example of an initial value problem with a solution that is not unique.

Solution 3.

$$y' = y^{1/3}, \quad y(0) = 0.$$

Problem 4. Find the general solution to the differential equation

$$y + \tan(x)y' = \sec^2(x)$$

Solution 4. An integrating factor for this equation is $\mu = \cos(x)$. Multiplying the original equation by this, we get the exact equation

$$\cos(x)y + \sin(x)y' = \sec(x).$$

The left-hand side may be replaced by $(\sin(x)y)'$, so that

$$(\sin(x)y)' = \sec(x).$$

Integrating both sides, we obtain

$$\sin(x)y = \ln|\sec(x) + \tan(x)| + C.$$

Therefore

$$y = \csc(x)\ln|\sec(x) + \tan(x)| + C\csc(x).$$

Problem 5. Initially, a tank contains 6 gal of water containing 1 lb of salt. There is water flowing into the tank through two pipes: Water containing salt is entering the tank through the first pipe at rate of 2 gal/min. Several measurements indicate that the amount of salt contained in one gallon of the incoming water is $e^{3t/2}$ lb at time t. One gallon of fresh water per minute is entering the tank through the second pipe. Finally, the well-stirred mixture is draining the tank at a rate of 3 gal/min. Determine the amount of salt at any time t.

Solution 5. The total volume of liquid entering the tank per unit time is equal to the volume leaving per unit time, and therefore the volume in the tank is a constant 6 gallons. Then rate in/rate out dictates that S satisfies the differential equation

$$\frac{dS}{dt} = 2e^{3t/2} - 3\frac{S}{6},$$

along with the initial condition $S(0) = 1$. An integrating factor for this equation is $\mu = e^{t/2}$. Multiplying by this integrating factor, and pulling all the S and S' terms to the left hand side, we get

$$\frac{1}{2}e^{t/2}S + e^{t/2}\frac{dS}{dt} = 2e^{-t}.$$

The left-hand side of this equation is equal to $(e^{t/2}S)'$, and therefore

$$(e^{t/2}S)' = 2e^{2t}.$$

Integrating both sides of this equation, we then find

$$e^{t/2}S = e^{2t} + C.$$

The initial condition implies $1 = 1 + C$, so that $C = 0$. Therefore

$$S = e^{3t/2}.$$