Math 307 Lecture 10

Second-Order Homogeneous Linear ODEs with Constant Coefficients

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Department of Mathematics
University of Washington

April 21, 2014

Today!

Last time:

- Return of Exact Equations and Integrating Factors!

This time:

- Higher-order Homogeneous Linear Equations with Constant Coefficients

Next time:

- Review for First Midterm

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Outline

1. Let's Understand the Title!
   - Second Order Linear Equations
   - Homogeneous Equations with Constant Coefficients
2. Solving 2nd-Order Linear ODEs with Const. Coeff.
   - The Method
   - An Example
   - Initial Value Problems

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   - Homogeneous Equations with Constant Coefficients

2. Solving 2nd-Order Linear ODEs with Const. Coeff.
   - The Method
   - An Example
   - Initial Value Problems

Second Order Linear Equations

- What is a linear ordinary differential equation of second order?

Definition

A second-order linear equation is an equation of the form

$$y'' + p(t)y' + q(t)y = f(t)$$

- For reasons that stem from physics, the function f is called the *forcing function*

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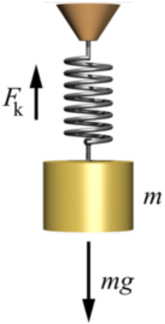
Example: An Ideal Mass-Spring System

Figure : A mass spring system can be modeled by a second-order equation



- Let x be effective length of spring (length - equilibrium length)
- Spring force: $F_{\text{spring}} = -kx$ with k the spring constant
- Grav force: $F_{\text{grav}} = mg$
- $F_{\text{total}} = F_{\text{spring}} + F_{\text{grav}}$
- Newton's Law: $F = mx''$
- Hence motion satisfies a second-order linear ODE

$$mx'' = -kx + mg = 0$$

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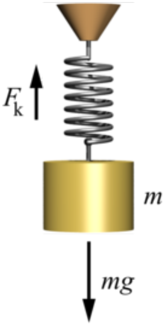

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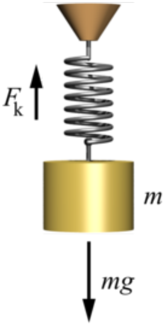

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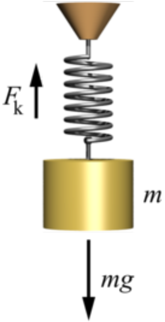

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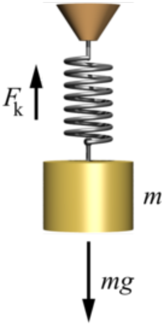

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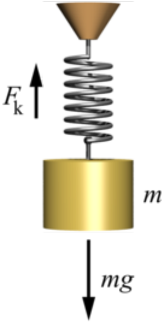

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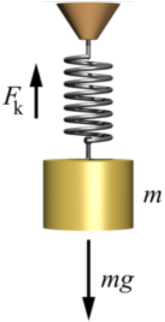

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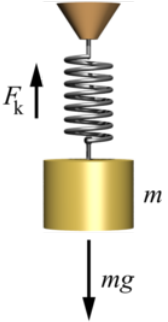

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Second-Order Linear Homogeneous Equations

- When is a Second-Order Linear Equation Homogeneous?

Definition

A second-order linear equation

$$y'' + p(t)y' + q(t)y = f(t)$$

is called homogeneous if $f(t) = 0$.

- For example:
- homogeneous: $y'' + xy' + \sin(x)y = 0$
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The Superposition Principle

- Homogeneous linear equations are nice, because they satisfy the “superposition principle":

Theorem (Superposition Principle)

If y_1 and y_2 are two solutions to a homogeneous linear ODE, then for any constants c_1 and c_2, the function $c_1 y_1 + c_2 y_2$ is also a solution

- For example:
- $\sin(x)$ and $\cos(x)$ are two solutions of the homogeneous linear ODE $y'' + y = 0$ (check this!!)
- By the superposition principal, $-13\sin(t) + 2\cos(t)$ is also a solution to the ODE (double check!!)

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Linear Equations with Constant Coefficients

- What does it mean for a second order linear ODE to have constant coefficients?

Definition

A second-order linear equation

$$y'' + p(t)y' + q(t)y = f(t)$$

has constant coefficients if p and q are constant.

- For example:
- $y'' + xy' + \sin(x)y = 0$ does NOT have const. coeff.
- $y'' + 3y' - 12y = \sec(x)$ does have const. coeff.

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Outline

How to Solve Hom. Linear ODEs with Const. Coeff

Figure : A Group of Dogs Solving ODEs by bluffing

- How do we solve second-order linear homogeneous ODEs with constant coefficients?
- We make a **crazy bluff** – that we already know the actual solution!
- We say that the solution is

$$y = e^{rt}$$

- How crazy is that?

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- But it works! If $y = e^{rt}$, then

$$y' = re^{rt}$$
$$y'' = r^2 e^{rt}$$

- Since y is a solution of our ODE, this means

$$\begin{aligned} 0 &= y'' + ay + b \\ &= r^2 e^{rt} + are^{rt} + be^{rt} \\ &= (r^2 + ar + b)e^{rt} \end{aligned}$$

- Hence r must be a root of the polynomial $x^2 + ax + b$.

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How to Solve Hom. Linear ODEs with Const. Coeff

- The polynomial $x^2 + ax + b$ is called the *characteristic polynomial* of the equation
- The polynomial $x^2 + ax + b$ will have two roots, say r_1 and r_2
- If the roots are **distinct**, then we will have **two** solutions $e^{r_1 t}$ and $e^{r_2 t}$
- By the **superposition principal**, we know

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

  will be a solution as well!
- In fact, this will be the general solution

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Our First Example

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Find the general solution of the second-order homogeneous linear ODE

$$y'' + 2y' - 3y = 0$$

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- This means $y' = re^{rt}$ and $y'' = r^2e^{rt}$

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Outline

IVP's for Second-Order Equations

- When it comes to second order linear equations, we've also got initial value problems
- However, since there are two arbitrary constants in our general solution, we need more information to specify the "initial condition"
- At the initial time t_0, we need to specify the initial value $y(t_0)$ and initial first derivative $y'(t_0)$
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$$y'' + y' - 2y = 0, \quad y(0) = 1, \; y'(0) = 1$$

- We propose a solution of the form $y = e^{rt}$.
- Then $y'' + y' - 2y = 0$ gives us the *characteristic equation*

$$r^2 + r - 2 = 0$$

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