Math 307 Lecture 11 Second-Order Homogeneous Linear ODEs with Constant Coefficients II

W.R. Casper

Department of Mathematics University of Washington

May 1, 2014

W.R. Casper [Math 307 Lecture 11](#page-105-0)

K ロ ト K 何 ト K ヨ ト K ヨ ト

÷.

Last time:

- Looked at 2nd-Order Hom. Lin. Eqns. with Constant Coefficients for the first time
- **Found out how to solve them in the case that the** characteristic polynomial had distinct roots
- This time:
	- Complex numbers
	- 2nd-Order Hom. Lin. Eqns. with Constant Coefficients with characteristic polynomials having complex roots

Next time:

2nd-Order Hom. Lin. Eqns. with Constant Coefficients with characteristic polynomials having repeated roots

≮ロト ⊀ 何 ト ⊀ ヨ ト ⊀ ヨ ト

Last time:

- Looked at 2nd-Order Hom. Lin. Eqns. with Constant Coefficients for the first time
- **Found out how to solve them in the case that the** characteristic polynomial had distinct roots
- This time:
	- Complex numbers
	- 2nd-Order Hom. Lin. Eqns. with Constant Coefficients with characteristic polynomials having complex roots

Next time:

2nd-Order Hom. Lin. Eqns. with Constant Coefficients with characteristic polynomials having repeated roots

4 ロ) (何) (日) (日)

 290

G

Last time:

- Looked at 2nd-Order Hom. Lin. Eqns. with Constant Coefficients for the first time
- **•** Found out how to solve them in the case that the characteristic polynomial had distinct roots

This time:

- Complex numbers
- 2nd-Order Hom. Lin. Eqns. with Constant Coefficients with characteristic polynomials having complex roots

Next time:

2nd-Order Hom. Lin. Eqns. with Constant Coefficients with characteristic polynomials having repeated roots

4 ロ) (何) (日) (日)

 290

B

Last time:

- Looked at 2nd-Order Hom. Lin. Eqns. with Constant Coefficients for the first time
- **•** Found out how to solve them in the case that the characteristic polynomial had distinct roots

This time:

- Complex numbers
- 2nd-Order Hom. Lin. Eqns. with Constant Coefficients with characteristic polynomials having complex roots

Next time:

2nd-Order Hom. Lin. Eqns. with Constant Coefficients with characteristic polynomials having repeated roots

4 ロ) (何) (日) (日)

 290

B

Last time:

- Looked at 2nd-Order Hom. Lin. Eqns. with Constant Coefficients for the first time
- **•** Found out how to solve them in the case that the characteristic polynomial had distinct roots

This time:

• Complex numbers

2nd-Order Hom. Lin. Eqns. with Constant Coefficients with characteristic polynomials having complex roots

Next time:

2nd-Order Hom. Lin. Eqns. with Constant Coefficients with characteristic polynomials having repeated roots

4 ロ) (何) (日) (日)

 290

B

Last time:

- Looked at 2nd-Order Hom. Lin. Eqns. with Constant Coefficients for the first time
- **•** Found out how to solve them in the case that the characteristic polynomial had distinct roots

This time:

- Complex numbers
- 2nd-Order Hom. Lin. Eqns. with Constant Coefficients with characteristic polynomials having complex roots

Next time:

2nd-Order Hom. Lin. Eqns. with Constant Coefficients with characteristic polynomials having repeated roots

4 ロ) (何) (日) (日)

B

Last time:

- Looked at 2nd-Order Hom. Lin. Eqns. with Constant Coefficients for the first time
- **•** Found out how to solve them in the case that the characteristic polynomial had distinct roots

This time:

- Complex numbers
- 2nd-Order Hom. Lin. Eqns. with Constant Coefficients with characteristic polynomials having complex roots

Next time:

2nd-Order Hom. Lin. Eqns. with Constant Coefficients with characteristic polynomials having repeated roots

4 ロ) (何) (日) (日)

B

Last time:

- Looked at 2nd-Order Hom. Lin. Eqns. with Constant Coefficients for the first time
- **•** Found out how to solve them in the case that the characteristic polynomial had distinct roots

This time:

- Complex numbers
- 2nd-Order Hom. Lin. Eqns. with Constant Coefficients with characteristic polynomials having complex roots

Next time:

2nd-Order Hom. Lin. Eqns. with Constant Coefficients with characteristic polynomials having repeated roots

イロト イ押 トイヨ トイヨト

[Complex Numbers](#page-10-0)

- **[Complex Number Basics](#page-10-0)**
- **•** [Euler's Definition](#page-44-0)
- 2 [Complex Roots of the Characteristic Polynomial](#page-67-0)
	- [General solutions to 2nd Order Linear ODEs with Const.](#page-67-0) [Coeff](#page-67-0)
	- **[General Case](#page-86-0)**

4 ロ) (何) (日) (日)

[Complex Number Basics](#page-10-0) [Euler's Definition](#page-44-0)

Outline

- [Complex Number Basics](#page-10-0)
- **•** [Euler's Definition](#page-44-0)
- 2 [Complex Roots of the Characteristic Polynomial](#page-67-0)
	- [General solutions to 2nd Order Linear ODEs with Const.](#page-67-0) [Coeff](#page-67-0)
	- [General Case](#page-86-0)

イロメ イ押 メイヨメ イヨメ

[Complex Number Basics](#page-10-0) [Euler's Definition](#page-44-0)

What are imaginary numbers?

- An *imaginary number* is any real number multiplied by [√] −1
- Usually denote this by *i*
- **examples:** 13*i*, 2*i*, −4*i*, π*i*

Figure : Imaginary numbers are very real to tigers

Calvin and Hobbes

by Bill Watterson

イロト イ押 トイヨ トイヨ トー

[Complex Number Basics](#page-10-0) [Euler's Definition](#page-44-0)

What are imaginary numbers?

- An *imaginary number* is any real number multiplied by $\sqrt{-1}$
- Usually denote this by *i*
- **examples:** 13*i*, 2*i*, −4*i*, π*i*

Figure : Imaginary numbers are very real to tigers

Calvin and Hobbes

by Bill Watterson

イロト イ押 トイヨ トイヨ トー

What are imaginary numbers?

- An *imaginary number* is any real number multiplied by $\sqrt{-1}$
- Usually denote this by *i* √
- **examples:** 13*i*, 2*i*, −4*i*, π*i*

Figure : Imaginary numbers are very real to tigers

Calvin and Hobbes

by Bill Watterson

≮ロ ▶ ⊀ 御 ▶ ⊀ ヨ ▶ ⊀ ヨ ▶

What are imaginary numbers?

- An *imaginary number* is any real number multiplied by $\sqrt{-1}$
- Usually denote this by *i*
- **examples:** 13*i*, √ 2*i*, −4*i*, π*i*

Figure : Imaginary numbers are very real to tigers

Calvin and Hobbes

by Bill Watterson

イロト イ押 トイヨ トイヨト

What are imaginary numbers?

- An *imaginary number* is any real number multiplied by $\sqrt{-1}$
- Usually denote this by *i*
- **examples:** 13*i*, √ 2*i*, −4*i*, π*i*

Figure : Imaginary numbers are very real to tigers

Calvin and Hobbes

by Bill Watterson

イロト イ押 トイヨ トイヨト

[Complex Number Basics](#page-10-0) [Euler's Definition](#page-44-0)

What are complex numbers?

where *a* and *b* are any real numbers.

- Any real number is a complex number also
- Any imaginary number is a complex number also

•
$$
2 + 3i
$$
, $-\sqrt{7} - \frac{1}{2}i$ and $4 + 2\pi i$ are complex numbers too

 290

K ロ ⊁ K 何 ≯ K ヨ ⊁ K ヨ ⊁

[Complex Number Basics](#page-10-0) [Euler's Definition](#page-44-0)

What are complex numbers?

Definition

A *complex number z* is a number of the form

where *a* and *b* are any real numbers.

- Any real number is a complex number also
- Any imaginary number is a complex number also

•
$$
2 + 3i
$$
, $-\sqrt{7} - \frac{1}{2}i$ and $4 + 2\pi i$ are complex numbers too

4 ロ) (何) (日) (日)

B

[Complex Number Basics](#page-10-0) [Euler's Definition](#page-44-0)

What are complex numbers?

Definition

A *complex number z* is a number of the form

where *a* and *b* are any real numbers.

- Any real number is a complex number also
- Any imaginary number is a complex number also
- 2 + 3*i*, − $\sqrt{7} - \frac{1}{2}$ $\frac{1}{2}$ *i* and 4 + 2 π *i* are complex numbers too

イロト イ伊 トイヨ トイヨ トー

 \Rightarrow

[Complex Number Basics](#page-10-0) [Euler's Definition](#page-44-0)

What are complex numbers?

Definition

where *a* and *b* are any real numbers.

- Any real number is a complex number also
- Any imaginary number is a complex number also √

2 + 3*i*, − $\sqrt{7}-\frac{1}{2}$ $\frac{1}{2}$ *i* and 4 + 2 π *i* are complex numbers too

モニー・モン イミン イヨン エミ

[Complex Number Basics](#page-10-0) [Euler's Definition](#page-44-0)

What are complex numbers?

Definition A *complex number z* is a number of the form $z = \int a^2 + b$ real part imaginary part \bigcap_{b} *b i* where *a* and *b* are any real numbers.

- Any real number is a complex number also
- Any imaginary number is a complex number also

• 2 + 3*i*,
$$
-\sqrt{7} - \frac{1}{2}i
$$
 and 4 + 2 πi are complex numbers too

イロメ 不優 トメ ヨ メ ス ヨ メー

 2990

÷.

Question

How do we add, subtract, multiply, and divide complex numbers by other complex numbers?

Addition and Subtraction:

- To add complex things, just add real and imaginary parts.
- For example

$$
(2+3i) + (4+2\pi i) = 6 + (3+2\pi)i
$$

• Similar story for subtraction...

$$
(2+3i) - (4+2\pi i) = -2 + (3-2\pi)i
$$

イロメ イ押 メイヨメ イヨメ

B

Question

How do we add, subtract, multiply, and divide complex numbers by other complex numbers?

Addition and Subtraction:

To add complex things, just add real and imaginary parts. • For example

$$
(2+3i) + (4+2\pi i) = 6 + (3+2\pi)i
$$

• Similar story for subtraction...

$$
(2+3i) - (4+2\pi i) = -2 + (3-2\pi)i
$$

イロメ イ押 メイヨメ イヨメ

B

Question

How do we add, subtract, multiply, and divide complex numbers by other complex numbers?

Addition and Subtraction:

To add complex things, just add real and imaginary parts.

• For example

$(2+3i) + (4+2\pi i) = 6 + (3+2\pi)i$

• Similar story for subtraction...

$$
(2+3i) - (4+2\pi i) = -2 + (3-2\pi)i
$$

イロト イ押 トイヨ トイヨ トー

Question

How do we add, subtract, multiply, and divide complex numbers by other complex numbers?

Addition and Subtraction:

- To add complex things, just add real and imaginary parts.
- For example

$$
(2+3i) + (4+2\pi i) = 6 + (3+2\pi)i
$$

• Similar story for subtraction...

$$
(2+3i) - (4+2\pi i) = -2 + (3-2\pi)i
$$

イロメ イ押 メイヨメ イヨメ

Question

How do we add, subtract, multiply, and divide complex numbers by other complex numbers?

Addition and Subtraction:

- To add complex things, just add real and imaginary parts.
- For example

$$
(2+3i) + (4+2\pi i) = 6 + (3+2\pi)i
$$

• Similar story for subtraction...

$$
(2+3i)-(4+2\pi i)=-2+(3-2\pi)i
$$

イロメ イ押 メイヨメ イヨメ

Question

How do we add, subtract, multiply, and divide complex numbers by other complex numbers?

Multiplication:

- To multiply complex things, we have to "foil", remembering that $i^2 = -1$
- For example

 $(2 + 3i) \cdot (4 + 2\pi i) = 2 \cdot 4 + 2 \cdot 2\pi i + 3i \cdot 4 + 3i \cdot 2\pi i$ $= 8 + 4\pi i + 12i + 6\pi i^2$ $= 8 + 4\pi i + 12i - 6\pi$ $= (8 - 6\pi) + (12 + 4\pi)i$

4 ロ) (何) (日) (日)

G

Question

How do we add, subtract, multiply, and divide complex numbers by other complex numbers?

Multiplication:

- To multiply complex things, we have to "foil", remembering that $i^2 = -1$
- For example

 $(2 + 3i) \cdot (4 + 2\pi i) = 2 \cdot 4 + 2 \cdot 2\pi i + 3i \cdot 4 + 3i \cdot 2\pi i$ $= 8 + 4\pi i + 12i + 6\pi i^2$ $= 8 + 4\pi i + 12i - 6\pi$ $= (8 - 6\pi) + (12 + 4\pi)i$

K ロ ト K 何 ト K ヨ ト K ヨ ト

B

Question

How do we add, subtract, multiply, and divide complex numbers by other complex numbers?

Multiplication:

To multiply complex things, we have to "foil", remembering that $i^2 = -1$

• For example

 $(2 + 3i) \cdot (4 + 2\pi i) = 2 \cdot 4 + 2 \cdot 2\pi i + 3i \cdot 4 + 3i \cdot 2\pi i$ $= 8 + 4\pi i + 12i + 6\pi i^2$ $= 8 + 4\pi i + 12i - 6\pi$ $= (8 - 6\pi) + (12 + 4\pi)i$

イロト イ団ト イヨト イヨト

B

 QQ

Question

How do we add, subtract, multiply, and divide complex numbers by other complex numbers?

Multiplication:

- To multiply complex things, we have to "foil", remembering that $i^2 = -1$
- **•** For example

$$
(2+3i) \cdot (4+2\pi i) = 2 \cdot 4 + 2 \cdot 2\pi i + 3i \cdot 4 + 3i \cdot 2\pi i
$$

= 8 + 4\pi i + 12i + 6\pi i²
= 8 + 4\pi i + 12i - 6\pi
= (8 - 6\pi) + (12 + 4\pi)i

イロト イ押 トイヨ トイヨ トー

Question

How do we add, subtract, multiply, and divide complex numbers by other complex numbers?

Inverses:

- We calculate the inverse of a complex number $z = a + bi$ by using the complex conjugate trick
- Complex conjugate is *z* [∗] = *a* − *bi*

$$
\frac{1}{a+bi} = \frac{1}{a+bi} \cdot 1 = \frac{1}{a+bi} \cdot \frac{(a-bi)}{(a-bi)} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i
$$

イロト イ押 トイヨ トイヨ トー

Question

How do we add, subtract, multiply, and divide complex numbers by other complex numbers?

Inverses:

• We calculate the inverse of a complex number $z = a + bi$ by using the complex conjugate trick

Complex conjugate is *z* [∗] = *a* − *bi*

$$
\frac{1}{a+bi} = \frac{1}{a+bi} \cdot 1 = \frac{1}{a+bi} \cdot \frac{(a-bi)}{(a-bi)}
$$

$$
= \frac{a-bi}{(a+bi)(a-bi)} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i
$$

イロト イ押 トイヨ トイヨ トー

Question

How do we add, subtract, multiply, and divide complex numbers by other complex numbers?

Inverses:

• We calculate the inverse of a complex number $z = a + bi$ by using the complex conjugate trick

Complex conjugate is *z* [∗] = *a* − *bi*

$$
\frac{1}{a+bi} = \frac{1}{a+bi} \cdot 1 = \frac{1}{a+bi} \cdot \frac{(a-bi)}{(a-bi)} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i
$$

イロト イ団ト イヨト イヨト

Question

How do we add, subtract, multiply, and divide complex numbers by other complex numbers?

Inverses:

- We calculate the inverse of a complex number $z = a + bi$ by using the complex conjugate trick
- Complex conjugate is *z* [∗] = *a* − *bi*

$$
\frac{1}{a+bi} = \frac{1}{a+bi} \cdot 1 = \frac{1}{a+bi} \cdot \frac{(a-bi)}{(a-bi)} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i
$$

イロト イ押 トイヨ トイヨト

Question

How do we add, subtract, multiply, and divide complex numbers by other complex numbers?

Inverses:

- We calculate the inverse of a complex number $z = a + bi$ by using the complex conjugate trick
- Complex conjugate is *z* [∗] = *a* − *bi*

$$
\frac{1}{a+bi} = \frac{1}{a+bi} \cdot 1 = \frac{1}{a+bi} \cdot \frac{(a-bi)}{(a-bi)} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i
$$

イロト イ押 トイヨ トイヨト

Question

How do we add, subtract, multiply, and divide complex numbers by other complex numbers?

Division:

- We divide by multiplying by the inverse
- Alternatively we apply complex conjugate trick
- For example

$$
\frac{4+2\pi i}{2+3i} = \frac{4+2\pi i}{2+3i} \cdot 1 = \frac{4+2\pi i}{2+3i} \cdot \frac{(2-3i)}{(2-3i)}
$$

$$
= \frac{(4+2\pi i) \cdot (2-3i)}{(2+3i) \cdot (2-3i)} = \frac{8+6\pi}{13} + \frac{4\pi - 12}{13}i
$$

イロメ イ押 メイヨメ イヨメ

B

 QQ
Question

How do we add, subtract, multiply, and divide complex numbers by other complex numbers?

Division:

- We divide by multiplying by the inverse
- Alternatively we apply complex conjugate trick
- For example

$$
\frac{4+2\pi i}{2+3i} = \frac{4+2\pi i}{2+3i} \cdot 1 = \frac{4+2\pi i}{2+3i} \cdot \frac{(2-3i)}{(2-3i)}
$$

$$
= \frac{(4+2\pi i) \cdot (2-3i)}{(2+3i) \cdot (2-3i)} = \frac{8+6\pi}{13} + \frac{4\pi - 12}{13}
$$

イロメ イ押 メイヨメ イヨメ

B

Question

How do we add, subtract, multiply, and divide complex numbers by other complex numbers?

Division:

- We divide by multiplying by the inverse
- Alternatively we apply complex conjugate trick
- For example

$$
\frac{4+2\pi i}{2+3i} = \frac{4+2\pi i}{2+3i} \cdot 1 = \frac{4+2\pi i}{2+3i} \cdot \frac{(2-3i)}{(2-3i)}
$$

$$
= \frac{(4+2\pi i) \cdot (2-3i)}{(2+3i) \cdot (2-3i)} = \frac{8+6\pi}{13} + \frac{4\pi - 12}{13}i
$$

イロト イ押 トイヨ トイヨ トー

B

Question

How do we add, subtract, multiply, and divide complex numbers by other complex numbers?

Division:

- We divide by multiplying by the inverse
- Alternatively we apply complex conjugate trick
- For example

$$
\frac{4+2\pi i}{2+3i} = \frac{4+2\pi i}{2+3i} \cdot 1 = \frac{4+2\pi i}{2+3i} \cdot \frac{(2-3i)}{(2-3i)}
$$

$$
= \frac{(4+2\pi i) \cdot (2-3i)}{(2+3i) \cdot (2-3i)} = \frac{8+6\pi}{13} + \frac{4\pi-12}{13}i
$$

イロメ イ押メ イヨメ イヨメー

B

Question

How do we add, subtract, multiply, and divide complex numbers by other complex numbers?

Division:

- We divide by multiplying by the inverse
- Alternatively we apply complex conjugate trick
- For example

$$
\frac{4+2\pi i}{2+3i} = \frac{4+2\pi i}{2+3i} \cdot 1 = \frac{4+2\pi i}{2+3i} \cdot \frac{(2-3i)}{(2-3i)}
$$

$$
= \frac{(4+2\pi i) \cdot (2-3i)}{(2+3i) \cdot (2-3i)} = \frac{8+6\pi}{13} + \frac{4\pi-12}{13}i
$$

イロメ イ押 メイヨメ イヨメ

B

[Complex Number Basics](#page-10-0) [Euler's Definition](#page-44-0)

Have a go at the following calculations:

イロト イ押 トイヨ トイヨ トー

 \bullet

 \bullet

[Complex Number Basics](#page-10-0) [Euler's Definition](#page-44-0)

Have a go at the following calculations:

$$
\frac{1+2i}{3-7i}=?
$$

メロメメ 御きメ 老き メ 悪き し

 \bullet

 \bullet

[Complex Number Basics](#page-10-0) [Euler's Definition](#page-44-0)

Have a go at the following calculations:

$(-3 + 2i) + (4 - 6i) = ?$

$$
(3+i)*(2-i) = ?
$$

$$
\frac{1+2i}{3-7i}=?
$$

イロト イ押 トイヨ トイヨ トー

 \bullet

 \bullet

 \bullet

Have a go at the following calculations:

$(-3 + 2i) + (4 - 6i) = ?$

$$
\frac{1+2i}{3-7i}=?
$$

メロメメ 御きメ 老き メ 悪き し

[Complex Number Basics](#page-10-0) [Euler's Definition](#page-44-0)

Outline

- **[Complex Number Basics](#page-10-0)**
- **•** [Euler's Definition](#page-44-0)
- 2 [Complex Roots of the Characteristic Polynomial](#page-67-0)
	- [General solutions to 2nd Order Linear ODEs with Const.](#page-67-0) [Coeff](#page-67-0)
	- [General Case](#page-86-0)

イロメ イ押 メイヨメ イヨメ

÷. QQ

Complex numbers as vectors

Figure : A complex number is a vector in the plane

- We can "visualize" a complex number $z = x + iy$ as a *vector*
- The tip of the vector is put at the point (*x*, *y*)
- The base of the vector is put at the origin
- Using polar coordinates, we can write

 $x = r \cos(\theta)$ $y = r \sin(\theta)$

≮ロト ⊀ 何 ト ⊀ ヨ ト ⊀ ヨ ト

[Complex Number Basics](#page-10-0) [Euler's Definition](#page-44-0)

Complex numbers as vectors

Figure : A complex number is a vector in the plane

- We can "visualize" a complex number $z = x + iy$ as a *vector*
- The tip of the vector is put at the point (*x*, *y*)
- The base of the vector is put at the origin
- Using polar coordinates, we can write

 $x = r \cos(\theta)$ $y = r \sin(\theta)$

≮ロト ⊀ 何 ト ⊀ ヨ ト ⊀ ヨ ト

 290

[Complex Number Basics](#page-10-0) [Euler's Definition](#page-44-0)

Complex numbers as vectors

Figure : A complex number is a vector in the plane

- We can "visualize" a complex number $z = x + iy$ as a *vector*
- The tip of the vector is put at the point (*x*, *y*)
- The base of the vector is put at the origin
- Using polar coordinates, we can write

 $x = r \cos(\theta)$ $y = r \sin(\theta)$

≮ロト ⊀ 何 ト ⊀ ヨ ト ⊀ ヨ ト

 290

[Complex Number Basics](#page-10-0) [Euler's Definition](#page-44-0)

Complex numbers as vectors

Figure : A complex number is a vector in the plane

- We can "visualize" a complex number $z = x + iy$ as a *vector*
- The tip of the vector is put at the point (*x*, *y*)
- The base of the vector is put at the origin
- Using polar coordinates, we can write

```
x = r \cos(\theta)y = r \sin(\theta)
```
≮ロト ⊀ 何 ト ⊀ ヨ ト ⊀ ヨ ト

 290

€

Complex numbers as vectors

Figure : A complex number is a vector in the plane

- We can "visualize" a complex number $z = x + iy$ as a *vector*
- The tip of the vector is put at the point (*x*, *y*)
- The base of the vector is put at the origin
- Using polar coordinates, we can write

$$
x = r \cos(\theta)
$$

$$
y = r \sin(\theta)
$$

K ロ ⊁ K 何 ≯ K ヨ ⊁ K ヨ ⊁

 2990

ă,

Definition

If θ is a real number, then we *define*

$$
e^{i\theta} = \cos(\theta) + i\sin(\theta).
$$

This definition does not break anything.

- Q: What do we mean by this?
- A: $e^{i\theta}e^{i\phi}=e^{i(\theta+\phi)}$
- Also, this makes sense in a power-series sort of way:

$$
\sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!} i = \cos(\theta) + \sin(\theta) i = e^{i\theta}
$$

Definition

If θ is a real number, then we *define*

$$
e^{i\theta} = \cos(\theta) + i\sin(\theta).
$$

Theorem

This definition does not break anything.

- Q: What do we mean by this?
- A: $e^{i\theta}e^{i\phi}=e^{i(\theta+\phi)}$
- Also, this makes sense in a power-series sort of way:

$$
\sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!} i = \cos(\theta) + \sin(\theta) i = e^{i\theta}
$$

Definition

If θ is a real number, then we *define*

$$
e^{i\theta} = \cos(\theta) + i\sin(\theta).
$$

Theorem

This definition does not break anything.

- **Q: What do we mean by this?**
- A: $e^{i\theta}e^{i\phi}=e^{i(\theta+\phi)}$
- Also, this makes sense in a power-series sort of way:

$$
\sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!} i = \cos(\theta) + \sin(\theta) i = e^{i\theta}
$$

イロメ イ押 メイヨメ イヨメ

B

Definition

If θ is a real number, then we *define*

$$
e^{i\theta} = \cos(\theta) + i\sin(\theta).
$$

Theorem

This definition does not break anything.

- **Q: What do we mean by this?**
- A: $e^{i\theta}e^{i\phi}=e^{i(\theta+\phi)}$
- Also, this makes sense in a power-series sort of way:

$$
\sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!} i = \cos(\theta) + \sin(\theta) i = e^{i\theta}
$$

イロメ イ押 メイヨメ イヨメ

B

Definition

If θ is a real number, then we *define*

$$
e^{i\theta} = \cos(\theta) + i\sin(\theta).
$$

Theorem

This definition does not break anything.

● Q: What do we mean by this?

• A:
$$
e^{i\theta} e^{i\phi} = e^{i(\theta + \phi)}
$$

Also, this makes sense in a power-series sort of way:

$$
\sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!} i = \cos(\theta) + \sin(\theta) i = e^{i\theta}
$$

Complex numbers as vectors

Figure : What does multiplication do to vectors?

- Take a complex number *x* + *iy*
- Convert (x, y) to polar: *r* = $\sqrt{x^2 + y^2}$ $\theta = \tan^{-1}(y/x)$

• Then
$$
x + iy = re^{i\theta}
$$
 also!

• If
$$
z = r_0 e^{is}
$$
 and $w = r_1 e^{it}$
then

$$
zw = r_0 r_1 e^{i(s+t)}
$$

4 ロ) (何) (日) (日)

B

 $2Q$

Complex numbers as vectors

Figure : What does multiplication do to vectors?

- Take a complex number *x* + *iy*
- Convert (x, y) to polar: *r* = $\sqrt{x^2 + y^2}$ $\theta = \tan^{-1}(y/x)$

• Then
$$
x + iy = re^{i\theta}
$$
 also!

• If
$$
z = r_0 e^{is}
$$
 and $w = r_1 e^{it}$
then

$$
zw = r_0 r_1 e^{i(s+t)}
$$

4 ロ) (何) (日) (日)

B

 $2Q$

Complex numbers as vectors

Figure : What does multiplication do to vectors?

- Take a complex number *x* + *iy*
- Convert (*x*, *y*) to polar: *r* = $\sqrt{x^2 + y^2}$ $\theta = \tan^{-1}(y/x)$

• Then
$$
x + iy = re^{i\theta}
$$
 also!

• If
$$
z = r_0 e^{is}
$$
 and $w = r_1 e^{it}$
then

$$
zw = r_0 r_1 e^{i(s+t)}
$$

4 ロ) (何) (日) (日)

B

 $2Q$

Complex numbers as vectors

Figure : What does multiplication do to vectors?

- Take a complex number *x* + *iy*
- Convert (*x*, *y*) to polar: *r* = $\sqrt{x^2 + y^2}$ $\theta = \tan^{-1}(y/x)$

• Then
$$
x + iy = re^{i\theta}
$$
 also!

If $z = r_0 e^{is}$ and $w = r_1 e^{it}$ then

$$
zw = r_0 r_1 e^{i(s+t)}
$$

4 ロ) (何) (日) (日)

ă,

 $2Q$

Complex numbers as vectors

Figure : What does multiplication do to vectors?

- Take a complex number *x* + *iy*
- Convert (*x*, *y*) to polar: *r* = $\sqrt{x^2 + y^2}$ $\theta = \tan^{-1}(y/x)$

• Then
$$
x + iy = re^{i\theta}
$$
 also!

If $z = r_0 e^{i s}$ and $w = r_1 e^{i t}$ then

$$
zw=r_0r_1e^{i(s+t)}
$$

4 ロ) (何) (日) (日)

 290

ă

A few more definitions

Figure : SMBC

• Let $z = re^{i\theta}$ be a complex number

- θ is called the *argument* of z
- *r* is called the *modulus* of *z* and is sometimes denoted as |*z*|
- easy exercise: show $r^2 = Z \cdot Z^*$
- Q: why do we care about complex numbers?
- A: simple! They give us more roots of polynomials!

イロト イ団ト イヨト イヨト

ă,

A few more definitions

Figure : SMBC

• Let $z = re^{i\theta}$ be a complex number

- θ is called the *argument* of z
- *r* is called the *modulus* of *z* and is sometimes denoted as |*z*|
- easy exercise: show $r^2 = Z \cdot Z^*$
- Q: why do we care about complex numbers?
- A: simple! They give us more roots of polynomials!

イロト イ団ト イヨト イヨト

ă

A few more definitions

Figure : SMBC

- Let $z = re^{i\theta}$ be a complex number
- θ is called the *argument* of z
- *r* is called the *modulus* of *z* and is sometimes denoted as |*z*|
- easy exercise: show $r^2 = Z \cdot Z^*$
- Q: why do we care about complex numbers?
- A: simple! They give us more roots of polynomials!

イロト イ団ト イヨト イヨト

ă,

A few more definitions

Figure : SMBC

- Let $z = re^{i\theta}$ be a complex number
- θ is called the *argument* of z
- *r* is called the *modulus* of *z* and is sometimes denoted as |*z*|
- easy exercise: show $r^2 = Z \cdot Z^*$
- Q: why do we care about complex numbers?
- A: simple! They give us more roots of polynomials!

イロト イ団ト イヨト イヨト

B

A few more definitions

Figure : SMBC

- Let $z = re^{i\theta}$ be a complex number
- θ is called the *argument* of z
- *r* is called the *modulus* of *z* and is sometimes denoted as |*z*|
- **e** easy exercise: show $r^2 = Z \cdot Z^*$
- Q: why do we care about complex numbers?
- A: simple! They give us more roots of polynomials!

イロト イ押 トイヨ トイヨ トー

B

A few more definitions

Figure : SMBC

- Let $z = re^{i\theta}$ be a complex number
- θ is called the *argument* of z
- *r* is called the *modulus* of *z* and is sometimes denoted as |*z*|
- **e** easy exercise: show $r^2 = Z \cdot Z^*$
- Q: why do we care about complex numbers?
- A: simple! They give us more roots of polynomials!

イロメ イ押メ イヨメ イヨメー

 290

A few more definitions

Figure : SMBC

- Let $z = re^{i\theta}$ be a complex number
- θ is called the *argument* of z
- *r* is called the *modulus* of *z* and is sometimes denoted as |*z*|
- **e** easy exercise: show $r^2 = Z \cdot Z^*$
- Q: why do we care about complex numbers?
- A: simple! They give us more roots of polynomials!

イロメ イ押メ イヨメ イヨメー

イロメ イ押 メイヨメ イヨメ

÷. QQ

Outline

[Complex Numbers](#page-10-0)

- **[Complex Number Basics](#page-10-0)**
- **•** [Euler's Definition](#page-44-0)
- 2 [Complex Roots of the Characteristic Polynomial](#page-67-0)
	- [General solutions to 2nd Order Linear ODEs with Const.](#page-67-0) [Coeff](#page-67-0)
	- [General Case](#page-86-0)

イロト イ団ト イヨト イヨト

÷. QQ

Our First Example

Example

Find the general solution of the second-order homogeneous linear ODE

$$
y''-2y'+2y=0
$$

- How do we find the general solution?
- First we bluff and say we already know our solution: $y=e^{rt}$.
- This means $y' = re^{rt}$ and $y'' = r^2e^{rt}$

イロト イ団ト イヨト イヨト

÷. QQ

Our First Example

Example

Find the general solution of the second-order homogeneous linear ODE

$$
y^{\prime\prime}-2y^{\prime}+2y=0
$$

• How do we find the general solution?

- First we bluff and say we already know our solution: $y=e^{rt}$.
- This means $y' = re^{rt}$ and $y'' = r^2e^{rt}$

イロト イ団ト イヨト イヨト

÷. QQ

Our First Example

Example

Find the general solution of the second-order homogeneous linear ODE

$$
y''-2y'+2y=0
$$

- How do we find the general solution?
- First we bluff and say we already know our solution: $y=e^{rt}$.
- This means $y' = re^{rt}$ and $y'' = r^2e^{rt}$

イロメ 不優 トメ ヨ メ ス ヨ メー

÷. QQ

Our First Example

Example

Find the general solution of the second-order homogeneous linear ODE

$$
y''-2y'+2y=0
$$

- How do we find the general solution?
- First we bluff and say we already know our solution: $y=e^{rt}$.
- This means $y' = re^{rt}$ and $y'' = r^2e^{rt}$
B

 $2Q$

Our First Example

$$
0 = y'' - 2y' + 2y
$$

= $r^2 e^{rt} - 2re^{rt} + 2e^{rt}$
= $(r^2 - 2r + 2)e^{rt}$

- Therefore $r^2 2r + 2 = 0$
- Roots of the polynomial $r^2 2r + 2$ are 1 ± *i*
- This means $y = e^{(1+i)t}$ and $y = e^{(1-i)t}$ are both solutions!
- Wait a minute, these are complex-valued functions!

B

 $2Q$

Our First Example

$$
0 = y'' - 2y' + 2y
$$

= $r^2e^{rt} - 2re^{rt} + 2e^{rt}$
= $(r^2 - 2r + 2)e^{rt}$

- Therefore $r^2 2r + 2 = 0$
- Roots of the polynomial $r^2 2r + 2$ are 1 ± *i*
- This means $y = e^{(1+i)t}$ and $y = e^{(1-i)t}$ are both solutions!
- Wait a minute, these are complex-valued functions!

B

 $2Q$

Our First Example

Then since *y* is a solution

$$
0 = y'' - 2y' + 2y
$$

= $r^2e^{rt} - 2re^{rt} + 2e^{rt}$
= $(r^2 - 2r + 2)e^{rt}$

Therefore $r^2 - 2r + 2 = 0$

- Roots of the polynomial $r^2 2r + 2$ are 1 ± *i*
- This means $y = e^{(1+i)t}$ and $y = e^{(1-i)t}$ are both solutions!
- Wait a minute, these are complex-valued functions!

B

 $2Q$

Our First Example

$$
0 = y'' - 2y' + 2y
$$

= $r^2e^{rt} - 2re^{rt} + 2e^{rt}$
= $(r^2 - 2r + 2)e^{rt}$

- Therefore $r^2 2r + 2 = 0$
- Roots of the polynomial $r^2 2r + 2$ are 1 \pm *i*
- This means $y = e^{(1+i)t}$ and $y = e^{(1-i)t}$ are both solutions!
- Wait a minute, these are complex-valued functions!

K ロ ト K 何 ト K ヨ ト K ヨ ト

B

 QQQ

Our First Example

$$
0 = y'' - 2y' + 2y
$$

= $r^2e^{rt} - 2re^{rt} + 2e^{rt}$
= $(r^2 - 2r + 2)e^{rt}$

- Therefore $r^2 2r + 2 = 0$
- Roots of the polynomial $r^2 2r + 2$ are 1 \pm *i*
- This means $y = e^{(1+i)t}$ and $y = e^{(1-i)t}$ are both solutions!
- Wait a minute, these are complex-valued functions!

K ロ ト K 何 ト K ヨ ト K ヨ ト

B

 QQQ

Our First Example

$$
0 = y'' - 2y' + 2y
$$

= $r^2 e^{rt} - 2re^{rt} + 2e^{rt}$
= $(r^2 - 2r + 2)e^{rt}$

- Therefore $r^2 2r + 2 = 0$
- Roots of the polynomial $r^2 2r + 2$ are 1 \pm *i*
- This means $y = e^{(1+i)t}$ and $y = e^{(1-i)t}$ are both solutions!
- Wait a minute, these are complex-valued functions!

 $2Q$

Our First Example

We only want real-valued solutions. So look at the general solution (obtained by superposition principal):

$$
y = Ae^{(1+i)t} + Be^{(1-i)t}
$$

• Using Euler's formula

 $y = Ae^{t}\cos(t) + iAe^{t}\sin(t) + Be^{t}\cos(t) - iBe^{t}\sin(t)$ $=(A+B)e^{t}\cos(t)+i(A-B)e^{t}\sin(t)$

K ロ ト K 何 ト K ヨ ト K ヨ ト

 $2Q$

Our First Example

We only want real-valued solutions. So look at the general solution (obtained by superposition principal):

$$
y = Ae^{(1+i)t} + Be^{(1-i)t}
$$

● Using Euler's formula

 $y = Ae^{t}\cos(t) + iAe^{t}\sin(t) + Be^{t}\cos(t) - iBe^{t}\sin(t)$ $=(A+B)e^{t}\cos(t)+i(A-B)e^{t}\sin(t)$

イロメ イ押 メイヨメ イヨメ

ă. QQQ

Our First Example

We only want real-valued solutions. So look at the general solution (obtained by superposition principal):

$$
y = Ae^{(1+i)t} + Be^{(1-i)t}
$$

Using Euler's formula

$$
y = Aet cos(t) + iAet sin(t) + Bet cos(t) - iBet sin(t)
$$

= (A + B)e^t cos(t) + i(A – B)e^t sin(t)

イロメ イ押 メイヨメ イヨメ

ă. QQQ

Our First Example

We only want real-valued solutions. So look at the general solution (obtained by superposition principal):

$$
y = Ae^{(1+i)t} + Be^{(1-i)t}
$$

● Using Euler's formula

$$
y = Aet cos(t) + iAet sin(t) + Bet cos(t) - iBet sin(t)
$$

= (A + B)e^t cos(t) + i(A – B)e^t sin(t)

メロメメ 御きメ ミカメ モド

÷.

 $2Q$

Our First Example

- \bullet Set $C = A + B$ and $D = i(A B)$
- Then *C* and *D* are arbitrary constants and the general solution is

$$
y = Ce^t \cos(t) + De^t \sin(t)
$$

÷.

 $2Q$

Our First Example

\bullet Set $C = A + B$ and $D = i(A - B)$

Then *C* and *D* are arbitrary constants and the general solution is

 $y = Ce^t \cos(t) + De^t \sin(t)$

メロメメ 御きメ ミカメ モド

÷.

 $2Q$

Our First Example

- \bullet Set $C = A + B$ and $D = i(A B)$
- Then *C* and *D* are arbitrary constants and the general solution is

$$
y = Ce^t \cos(t) + De^t \sin(t)
$$

イロメ イ押メ イヨメ イヨメー

÷.

 $2Q$

Our First Example

- \bullet Set $C = A + B$ and $D = i(A B)$
- Then *C* and *D* are arbitrary constants and the general solution is

$$
y = Ce^t \cos(t) + De^t \sin(t)
$$

イロメ イ押 メイヨメ イヨメ

÷. QQQ

Outline

[Complex Numbers](#page-10-0)

- **[Complex Number Basics](#page-10-0)**
- **•** [Euler's Definition](#page-44-0)

2 [Complex Roots of the Characteristic Polynomial](#page-67-0)

- [General solutions to 2nd Order Linear ODEs with Const.](#page-67-0) [Coeff](#page-67-0)
- **[General Case](#page-86-0)**

K ロメ K 御 メ K 君 メ K 君 X

ă.

 $2Q$

General Case

$$
y'' + ay' + by = 0
$$

- The corresponding characteristic polynomial will be $r^2 + ar + b$
- Roots come in conjugate pairs
- Suppose $r^2 + ar + b$ has the root $\alpha + \beta b$
- Then it will also have the root $\alpha \beta i$
- Therefore the general (complex-valued) solution will look like

$$
y = Ae^{(\alpha+i\beta)t} + Be^{(\alpha-i\beta)t}
$$

≮ロト ⊀ 何 ト ⊀ ヨ ト ⊀ ヨ ト

B

 $2Q$

General Case

$$
y^{\prime\prime}+ay^{\prime}+by=0
$$

- The corresponding characteristic polynomial will be $r^2 + ar + b$
- Roots come in conjugate pairs
- Suppose $r^2 + ar + b$ has the root $\alpha + \beta b$
- Then it will also have the root $\alpha \beta i$
- Therefore the general (complex-valued) solution will look like

$$
y = Ae^{(\alpha + i\beta)t} + Be^{(\alpha - i\beta)t}
$$

B

 QQQ

General Case

$$
y^{\prime\prime}+ay^{\prime}+by=0
$$

- The corresponding characteristic polynomial will be $r^2 + ar + b$
- Roots come in conjugate pairs
- Suppose $r^2 + ar + b$ has the root $\alpha + \beta b$
- Then it will also have the root $\alpha \beta i$
- Therefore the general (complex-valued) solution will look like

$$
y = Ae^{(\alpha+i\beta)t} + Be^{(\alpha-i\beta)t}
$$

B

 QQQ

General Case

$$
y^{\prime\prime}+ay^{\prime}+by=0
$$

- The corresponding characteristic polynomial will be $r^2 + ar + b$
- Roots come in conjugate pairs
- Suppose $r^2 + ar + b$ has the root $\alpha + \beta b$
- Then it will also have the root $\alpha \beta i$
- Therefore the general (complex-valued) solution will look like

$$
y = Ae^{(\alpha+i\beta)t} + Be^{(\alpha-i\beta)t}
$$

B

 QQQ

General Case

$$
y^{\prime\prime}+ay^{\prime}+by=0
$$

- The corresponding characteristic polynomial will be $r^2 + ar + b$
- Roots come in conjugate pairs
- Suppose $r^2 + ar + b$ has the root $\alpha + \beta b$
- Then it will also have the root $\alpha \beta i$
- Therefore the general (complex-valued) solution will look like

$$
y = Ae^{(\alpha+i\beta)t} + Be^{(\alpha-i\beta)t}
$$

B

 QQQ

General Case

$$
y'' + ay' + by = 0
$$

- The corresponding characteristic polynomial will be $r^2 + ar + b$
- Roots come in conjugate pairs
- Suppose $r^2 + ar + b$ has the root $\alpha + \beta b$
- Then it will also have the root $\alpha \beta i$
- Therefore the general (complex-valued) solution will look like

$$
y = Ae^{(\alpha + i\beta)t} + Be^{(\alpha - i\beta)t}
$$

イロメ 不優 トイヨメ イヨメー

÷. QQQ

General Case

$$
y^{\prime\prime}+ay^{\prime}+by=0
$$

- The corresponding characteristic polynomial will be $r^2 + ar + b$
- Roots come in conjugate pairs
- Suppose $r^2 + ar + b$ has the root $\alpha + \beta b$
- Then it will also have the root $\alpha \beta i$
- Therefore the general (complex-valued) solution will look like

$$
y = Ae^{(\alpha+i\beta)t} + Be^{(\alpha-i\beta)t}
$$

イロト イ団 トイヨ トイヨ トー

÷.

 $2Q$

General Case

• Euler's formula will then give

$$
y = (A + B)e^{\alpha t}\cos(\beta t) + (A - B)e^{\alpha t}\sin(\beta t)i
$$

• Setting $C = A + B$ and $D = (A - B)i$, the (real) general solution is

$$
y = Ce^{\alpha t} \cos(\beta t) + De^{\alpha t} \sin(\beta t)
$$

イロト イ団 トイヨ トイヨ トー

÷. QQQ

General Case

• Euler's formula will then give

$$
y = (A + B)e^{\alpha t}\cos(\beta t) + (A - B)e^{\alpha t}\sin(\beta t)i
$$

• Setting $C = A + B$ and $D = (A - B)i$, the (real) general solution is

$$
y = Ce^{\alpha t} \cos(\beta t) + De^{\alpha t} \sin(\beta t)
$$

メロトメ 御 トメ 差 トメ 差 トー

重。 QQQ

General Case

• Euler's formula will then give

$$
y = (A + B)e^{\alpha t}\cos(\beta t) + (A - B)e^{\alpha t}\sin(\beta t)i
$$

 \bullet Setting $C = A + B$ and $D = (A - B)i$, the (real) general solution is

$$
y = Ce^{\alpha t} \cos(\beta t) + De^{\alpha t} \sin(\beta t)
$$

イロト イ団 トイヨ トイヨ トー

ミー QQQ

General Case

• Euler's formula will then give

$$
y = (A + B)e^{\alpha t}\cos(\beta t) + (A - B)e^{\alpha t}\sin(\beta t)i
$$

• Setting $C = A + B$ and $D = (A - B)i$, the (real) general solution is

$$
y = Ce^{\alpha t} \cos(\beta t) + De^{\alpha t} \sin(\beta t)
$$

イロト イ押 トイヨ トイヨ トー

重し 2990

Try it Yourself!

 \bullet

 \bullet

 \bullet

 $y'' + 2y' - 8y = 0$

 $y'' + 6y' + 13y = 0$

 $y'' + 2y' + 1.25y = 0$

イロメ 不優 おす 重 おす 悪 おし

重。 299

Try it Yourself!

 \bullet

 \bullet

 \bullet

$$
y'' + 2y' - 8y = 0
$$

$$
y''+6y'+13y=0
$$

$$
y'' + 2y' + 1.25y = 0
$$

イロメ 不優 トイヨメ イヨメー

重。 299

Try it Yourself!

 \bullet

 \bullet

 \bullet

$$
y'' + 2y' - 8y = 0
$$

$$
y^{\prime\prime}+6y^{\prime}+13y=0
$$

$$
y'' + 2y' + 1.25y = 0
$$

イロト 不優 トイモト 不思 トー

重し 2990

Try it Yourself!

 \bullet

 \bullet

 \bullet

$$
y'' + 2y' - 8y = 0
$$

$$
y''+6y'+13y=0
$$

$$
y'' + 2y' + 1.25y = 0
$$

B

 $2Q$

Review!

Today:

- **Complex numbers and Euler's definition**
- What happens when the characteristic polynomial has complex roots

Next time:

What happens when the characteristic polynomial has repeated roots

÷.

 $2Q$

Review!

Today:

- Complex numbers and Euler's definition
- What happens when the characteristic polynomial has complex roots

Next time:

What happens when the characteristic polynomial has repeated roots

÷. QQQ

Review!

Today:

- Complex numbers and Euler's definition
- What happens when the characteristic polynomial has complex roots
- Next time:
	- What happens when the characteristic polynomial has repeated roots

÷. QQ

Review!

Today:

- Complex numbers and Euler's definition
- What happens when the characteristic polynomial has complex roots

Next time:

What happens when the characteristic polynomial has repeated roots