Math 307 Lecture 14

More on the Method of Undetermined Coefficients

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Today!

Last time:

- 2nd-Order Nonhomogeneous Linear ODEs. with Constant Coefficients

This time:

- More on 2nd-Order Nonhomogeneous Linear ODEs. with Constant Coefficients

Next time:

- Mechanical and Electrical Vibrations

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1. More on the Method of Undetermined Coefficients
   - Exception involving a solution to the homogeneous eqn.
   - Exception involving a (repeated) solution to the homogeneous eqn.
2. Solving other Inhomogeneous Equations
   - A few good examples
   - Try it Yourself

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An Example where Something Goes Wrong...

Example

Find a particular solution of the equation

$$y'' - 3y' - 4y = 3e^{4t}.$$

- How do we find a particular solution? What does experience suggest?
- Try a solution of the form $y = Ae^{4t}$, and figure out what A has to be.
- Notice that in this case $y' = 4Ae^{4t}$ and $y'' = 16Ae^{4t}$

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An Example where Something Goes Wrong...

- Plugging this back into the original ODE:

$$y'' - 3y' - 4y = 16Ae^{4t} - 12Ae^{4t} - 4Ae^{4t} = 0.$$

- Wait, $0 \neq 3e^{4t}$, so $y = Ae^{4t}$ cannot be a solution for any A
- What went wrong?
- The Ae^{4t} was a solution of the homogeneous ODE!
- This is TERRIBLE! What can we do to fix it?

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Let's Fix It!

- Try instead a solution of the form $y = Ate^{4t}$
- In this case,

$$y' = (4At + A)e^{4t}$$
$$y'' = (16At + 8A)e^{4t}$$

- One may then calculate

$$y'' - 3y' - 4y = 5Ae^{4t}$$

- Since $y'' - 3y' - 4y = 3e^{4t}$ was our original equation, this tells us $A = 3/5$.
- So our particular solution is $y = \frac{3}{5}e^{4t}$

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- This won't work! Why not?

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- Fine then, try $y = Ate^t$ instead
- This won't work either! Why not?
- Well...crap. What should we do?
- Try something of the form $y = At^2e^t$, maybe?
- YES! Note that

$$y' = A(t^2 + 2t)e^t$$
$$y'' = A(t^2 + 4t + 2)e^t$$

- So that (after some algebra)

$$y'' - 2y' + y = 2Ae^t.$$

- If we take $A = 3/2$, then we get a solution!

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Find a particular solution of the equation

$$y'' - 3y' - 4y = te^{4t}.$$

- What might we try?
- We could try $y = (At + B)e^{4t}$
- Won't work! Try it and see!
- Why didn't it work?
- Because e^{4t} is a solution of the homogeneous equation!
- Instead, try $y = (At^2 + Bt)e^{4t}$

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$$y'' - 3y' - 4y = (13t^2 - 7t + 8)e^{2t}.$$

- What might we try?
- We should try $y = (At^2 + Bt + C)e^{2t}$
- Try it and see!
- What should we change if instead the on the right hand side we have $(13t^2 - 7t + 8)e^{4t}$?
- Instead, try $(At^3 + Bt^2 + Ct)e^{4t}$

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- $y'' - 2y' - 3y = 3e^{3t}$
- $y'' - 2y' - 3y = e^{-t}\sin(t)$
- $y'' - 2y' - 3y = e^{-t}\cos(t)$
- $y'' - 2y' - 3y = 4te^{3t}$
- $y'' - 2y' - 3y = (4t - 6)e^{3t} + 2e^{-t}\sin(t) - e^{-t}\cos(t)$
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