Math 307 Quiz 5

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Problem 1. Find the general solution to the differential equation

$$y'' - 5y' + 6y = 3e^t$$

Solution 1. We first find the general solution to the corresponding homogeneous equation

$$y_h'' - 5y_h' + 6y_h = 0.$$

The characteristic polynomial of this equation is $r^2 - 5r + 6$, which has roots $2, 3$. Therefore the general solution to the homogeneous equation is

$$y_h(t) = C_1 e^{2t} + C_2 e^{3t}.$$

We next find a particular solution to the original homogeneous equation. We propose a solution of the form $y_p = Ae^t$. Then $y_p' = Ae^t$ and $y_p'' = Ae^t$, so that

$$y_p'' - 5y_p' + 6y_p = 2Ae^t.$$

Since we want $y_p'' - 5y_p' + 6y_p = 3e^t$, we should take $A = 3/2$. Then $y_p = \frac{3}{2}e^t$ is a particular solution, so the general solution is

$$y = y_h + y_p = C_1 e^{2t} + C_2 e^{3t} + \frac{3}{2}e^t.$$

Problem 2. Consider the differential equation

$$(*) \quad y'' - 5y' + 6y = 7e^{2t}\sin(t).$$

Suppose also $\widetilde{y}_p$ is a particular solution to the differential equation

$$\widetilde{y}'' - 5\widetilde{y}' + 6\widetilde{y} = 7e^{at}$$

for what value of a is $y_p = \mathrm{Im}(\widetilde{y}_p)$ a particular solution to $(*)$?

Solution 2. In order for $y_p = \text{Im}(\widetilde{y}_p)$ to be a particular solution to $(*)$, we need for

$$\text{Im}\left(7e^{at}\right) = 7e^{2t}\sin(t).$$

This is exactly the case if we take $a = 2 + i$.

Problem 3. Find the general solution to the differential equation

$$y'' - 5y' + 6y = 7e^{2t}\sin(t).$$

Solution 3. In the first problem, we found the general solution to the corresponding homogeneous equation

$$y_h(t) = C_1e^{2t} + C_2e^{3t}.$$

Thus we need only find a particular solution. To do so, we instead find a particular solution $\widetilde{y}_p$ of

$$\widetilde{y}_p'' - 5\widetilde{y}_p' + 6\widetilde{y}_p = 7e^{(2+i)t}.$$

To do so, we propose a solution of the form $\widetilde{y}_p = Ae^{(2+i)t}$ so that $\widetilde{y}_p' = (2+i)Ae^{(2+i)t}$ and $\widetilde{y}_p'' = (2+i)^2Ae^{(2+i)t}$, and therefore

$$\widetilde{y}_p'' - 5\widetilde{y}_p' + 6\widetilde{y}_p = \left[(2+i)^2 - 5(2+i) + 6\right]Ae^{(2+i)t} = (-1-i)Ae^{(2+i)t}.$$

Therefore to get a particular solution to our differential equation, we should take

$$A = 7/(-1-i) = -\frac{7}{2} + \frac{7}{2}i$$

Thus we get a particular solution to the $\sim$ equation

$$\widetilde{y}_p = \left(-\frac{7}{2} + \frac{7}{2}i\right)e^{(2+i)t} = -\frac{7}{2}e^{2t}\cos(t) - \frac{7}{2}e^{2t}\sin(t) + \left(\frac{7}{2}e^{2t}\cos(t) - \frac{7}{2}e^{2t}\sin(t)\right)i.$$

From this we see that a particular solution y_p to the original equation is given by

$$y_p = \text{Im}(\widetilde{y}_p) = \frac{7}{2}e^{2t}\cos(t) - \frac{7}{2}e^{2t}\sin(t)$$

Therefore the general solution is

$$y = y_h + y_p = C_1e^{2t} + C_2e^{3t} + \frac{7}{2}e^{2t}\cos(t) - \frac{7}{2}e^{2t}\sin(t)$$

Problem 4. Find the general solution to the differential equation

$$y'' - 5y' + 6y = e^{2t}\sin(t) + 4e^t.$$

Solution 4. Let y_1 be the particular solution we found in Problem 1, and y_2 be the particular solution we found in Problem 3. We define a linear operator L by

$$L[y] := y'' - 5y' + 6y.$$

The differential equation above may then be rewritten as

$$L[y] = e^{2t}\sin(t) + 4e^t.$$

Notice that $L[y_1] = 3e^t$ and $L[y_3] = 7e^{2t}\sin(t)$. Therefore, if A and B are any constants,

$$L[Ay_1 + By_2] = AL[y_1] + BL[y_2] = 3Ae^t + 7Be^{2t}\sin(t).$$

Thus if we take $A = 4/3$ and $B = 1/7$, then

$$L[(4/3)y_1 + (1/7)y_2] = e^{2t}\sin(t) + 4e^t.$$

Thus $y_p = (4/3)y_1 + (1/7)y_2$ is a particular solution to the above equation. Using the values of y_1 and y_2 obtained previously, we find our particular solution to be

$$y_p = 2e^t + \frac{1}{2}e^{2t}\cos(t) - \frac{1}{2}e^{2t}\sin(t).$$

Our general solution is therefore

$$y = y_h + y_p = C_1e^{2t} + C_2e^{3t} + \frac{1}{2}e^{2t}\cos(t) - \frac{1}{2}e^{2t}\sin(t).$$