Math 307 Quiz 5

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Problem 1. Find the general solution to the differential equation

$$y'' - 2y' + y = 3e^t$$

Solution 1. The corresponding homogeneous equation is

$$y_h'' - 2y_h' + y_h = 0.$$

The characteristic polynomial of this equation is $z^2 - 2z + 1$, which factors as $(z-1)^2$. Therefore the general solution is

$$y_h = (C_1 t + C_2)e^t.$$

The particular solution that we'd usually try for the forcing function $3e^t$ is $y_p = Ae^t$. However, since 1 is a root of the characteristic polynomial, with multiplicity 2, we will instead try

$$y_p = At^2e^t.$$

We then calculate

$$y_p' = (At^2 + 2At)e^t$$
$$y_p'' = (At^2 + 4At + 2A)e^t$$

so that

$$y_p'' - 2y_p' + y_p = 2Ae^t.$$

Therefore for y_p to be a particular solution to our original equation, we should take $A = 3/2$. Thus a particular solution is $y_p = (3/2)t^2e^t$ and our general solution is

$$y = y_h + y_p = (C_1 t + C_2)e^t + \frac{3}{2}t^2e^t.$$

Problem 2. Consider the differential equation

$$(*) \quad y'' - 2y' + y = 3e^t \sin(2t).$$

Suppose that $\widetilde{y}_p$ is a particular solution to the differential equation

$$\widetilde{y}'' - 2\widetilde{y}' + \widetilde{y} = 3e^{at}$$

for what value of a is $y_p = \text{Im}(\widetilde{y}_p)$ a particular solution to $(*)$?

Solution 2. Take $a = 1 + 2i$.

Problem 3. Find a particular solution to the differential equation

$$y'' - 2y' + y = 3e^t \sin(2t).$$

Solution 3. As pointed out in Problem 2, a particular solution y_p to the equation

$$y_p'' - 2y_p' + y_p = 3e^t \sin(2t).$$

may be found by taking the imaginary part of a particular solution $\widetilde{y}_p$ to the equation

$$\widetilde{y}_p'' - 2\widetilde{y}_p' + \widetilde{y}_p = 3e^{(1+2i)t}.$$

The method of undetermined coefficients tells us to try a particular solution of the form

$$\widetilde{y}_p = Ae^{(1+2i)t}.$$

We then calculate

$$\widetilde{y}_p' = (1 + 2i)Ae^{(1+2i)t}$$

and

$$\widetilde{y}_p'' = (1 + 2i)^2 Ae^{(1+2i)t} = (-3 + 4i)Ae^{(1+2i)t}.$$

so that

$$\widetilde{y}_p'' - 2\widetilde{y}_p' + \widetilde{y}_p = -4Ae^{(1+2i)t}.$$

Therefore for $\widetilde{y}_p$ to be a particular solution to the squiggly equation, we should take $A = -3/4$. Thus

$$\begin{aligned} \widetilde{y}_p &= \frac{-3}{4} e^{(3+2i)t} \\ &= \frac{-3}{4} e^t e^{2it} \\ &= \frac{-3}{4} e^t \cos(2t) - i\frac{3}{4} \sin(2t) \end{aligned}$$

Therefore a particular solution y_p to the original equation is given by

$$y_p = \mathrm{Im}(\widetilde{y}_p) = -\frac{3}{4}e^t \sin(2t).$$

Problem 4. Find a particular solution to the differential equation

$$y'' - 2y' + y = 4e^t \sin(2t) + 5e^t \cos(2t)$$

Solution 4. Let $\widetilde{y}_p$ be the function found in the solution to the previous problem. Then $\widetilde{y}_p$ is a solution to

$$\widetilde{y}_p'' - 2\widetilde{y}_p' + \widetilde{y}_p = 3e^{(1+2i)t}.$$

Therefore $(4/3)\widetilde{y}_p$ is a solution to

$$\widetilde{y}_p'' - 2\widetilde{y}_p' + \widetilde{y}_p = 4e^{(1+2i)t},$$

so that $y_1 = \mathrm{Im}((4/3)\widetilde{y}_p)$ is a solution to

$$y_1'' - 2y_1' + y_1 = 4e^t \sin(2t).$$

Similarly $(5/3)\widetilde{y}_p$ is a solution to

$$\widetilde{y}_p'' - 2\widetilde{y}_p' + \widetilde{y}_p = 5e^{(1+2i)t},$$

so that $y_2 = \mathrm{Re}((5/3)\widetilde{y}_p)$ is a solution to

$$y_1'' - 2y_1' + y_1 = 5e^t \cos(2t).$$

It follows that $y_p = y_1 + y_2$ is a solution to

$$y_p'' - 2y_p' + y_p = 4e^t \sin(2t) + 5e^t \cos(2t).$$

In summary, the particular solution we are looking for is

$$y_p = \frac{4}{3}\mathrm{Im}(\widetilde{y}_p) + \frac{5}{3}\mathrm{Re}(\widetilde{y}_p) = -e^t \sin(2t) - \frac{5}{4}e^t \cos(2t).$$