Math 307 Lecture 17

Forced Vibrations

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University of Washington

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Today!

Last time:

- More on Mechanical and Electrical Vibrations

This time:

- Forced vibrations

Next time:

- Review for final exam

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Outline

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What is Forced Vibration?

- Forced vibration is what occurs when we have an equation of the form
$$ay'' + by' + cy = R\cos(\omega t).$$
- where a, b, c are positive constants
- Intuitively, we should think about a spring with an external periodic force applied.
- What solutions look like depend very strongly on the values of a, b, c, R, and ω
- Today we will look at behavior when $b > 0$
- Later we will study separately the case when $b = 0$

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Examples of Forced Vibration

- Forced vibrations occur naturally all the time!
- For example one could think of
  - A mass-spring system, with an external force applied
  - A LCR-circuit, where the voltage input $E(t)$ is not constant
- What does forced vibration look like in comparison to unforced vibration?
- How does forced vibration behave at large timescales?

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LCR-circuit example

- Last class, we found that the current in an LCR-circuit with inductance L, capacitance C, and resistance R (all positive constants) satisfies the second-order equation

$$LI'' + RI' + I/C = E'(t)$$

- where here $E(t)$ is the voltage input into the LCR circuit by some power source
- power source could be a battery, or similar device
- could also be power from an electrical outlet or a motor with or without an alternator

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LCR-circuit example

- Suppose that our voltage source is V_0 volts
- Depending on the kind of source, this may mean two different things
- If the source is a battery, then the voltage is *direct* and $E(t) = V_0$, so that $E'(t) = 0$
- If the source is a wall outlet or alternator, the voltage is *alternating*
- In this case V_0 typically refers to the *rms voltage*, so that the true voltage input is $E(t) = \sqrt{2}V_0 \sin(\omega t)$, where ω is the electrical angular frequency of the power source
- For wall outlets in the U.S.A., $\omega/2\pi = 60$ Hz

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LCR-circuit: direct current case

- Suppose that our voltage source is V_0 volts in direct current (ie. from a battery or DC generator)
- Then $E(t)$ is constant, so $E'(t) = 0$
- Also assume that $R > 0$ (which is *always* true in real life)
- The current satisfies the equation

$$LI'' + RI' + I/C = 0$$

- and for $R^2 < 4L/C$, the solution is of the form

$$I = re^{-Rt/2L}\cos(\mu t - \delta)$$

- where $\mu = (\sqrt{R^2 - 4L/C})/2L$, and r, δ are constants depending on the initial conditions $I(0)$ and $I'(0)$

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- How does $I(t)$ behave at large time?
- It becomes very small, approaching zero exponentially fast
- Comparatively, the behavior of the current in the alternating voltage source case is much more interesting

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LCR-circuit: alternating current case

- Suppose that our voltage source is V_0 volts in alternating current (ie. from a wall socket or alternator)
- Then $E(t) = \sqrt{2}V_0 \sin(\omega t)$, so $E'(t) = \sqrt{2}\omega V_0 \cos(\omega t)$
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- The current satisfies the equation

$$LI'' + RI' + I/C = \sqrt{2}\omega V_0 \cos(\omega t)$$

- In this case, the solution has two components

$$I = \overbrace{c_1 I_1 + c_2 I_2}^{\text{solution to homogeneous equation}} + \overbrace{I_p}^{\text{particular solution}}$$

- Where c_1, c_2 are constants, depending on initial condition

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LCR-circuit: alternating current case

- As the solution to the homogeneous equation we derived before shows us, the solution to the homogeneous equation will become very small at large times
- Therefore, at large times, the general solution will look very similar to the particular solution
- For this reason, we often call $c_1 I_1 + c_2 I_2$ the *transient solution* (in the case of circuits, also called a transient current
- And we call I_p the *steady state solution* or *forced response* (in the circuit case, also called *steady state current*).

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- Thus we write

$$I = I_{\text{transient}} + I_{\text{steady}},$$

- where I_{steady} is a particular solution of

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- and where $I_{\text{transient}}$ is a solution of the corresponding homogeneous equation
$$LI'' + RI' + I/C = 0$$
- which we know get exponentially smaller as time increases when the resistance R is not zero

LCR-circuit: alternating current case

- Straightforward but tedious algebraic calculation may be used to show

$$I_{\text{steady}} = r\cos(\omega t - \delta),$$

- where

$$r = \frac{\sqrt{2}\omega V_0}{\Delta}, \quad \cos(\delta) = \frac{L(\omega_0^2 - \omega^2)}{\Delta}, \quad \sin(\delta) = \frac{R\omega}{\Delta}$$

- and also

$$\Delta = \sqrt{L^2(\omega_0^2 - \omega^2)^2 + R^2\omega^2}, \quad \omega_0 = 1/\sqrt{LC}$$

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Outline

LCR-circuit example

Example

Suppose that an LCR-circuit has a 1 H inductor, a 1 F capacitor, and a 0.125Ω resistor. Also suppose that the initially $I'(0) = 0$ and $I(0) = 2$. If a voltage source of 0.25 volts is connected, determine the current as a function of time.

We consider two cases:

- when the voltage source is direct current (DC)
- when the voltage source is alternating current (AC)

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1. Forced Vibrations Introduction
   - First Examples
   - Alternating Current and LCR Circuits
2. Forced Vibration Example
   - LCR Circuit Specific Example
   - DC Case
   - AC Case

LCR-circuit example: DC case

- Assume that the voltage source is direct
- We have that $C = 1$, $L = 1$, and $R = 0.125$.
- In this case, $E(t) = 0.25$, $E'(0) = 0$, so that I is a solution to the initial value problem

$$I'' + 0.125I' + I = 0, \;\; I(0) = 2, \;\; I'(0) = 0$$

- We solved this last time, and got

$$I = \frac{32}{\sqrt{255}} e^{-t/16} \cos\left(\frac{\sqrt{255}}{16} t - \delta\right)$$

for $\delta = \tan^{-1} \frac{1}{\sqrt{255}} \approx 0.0625$

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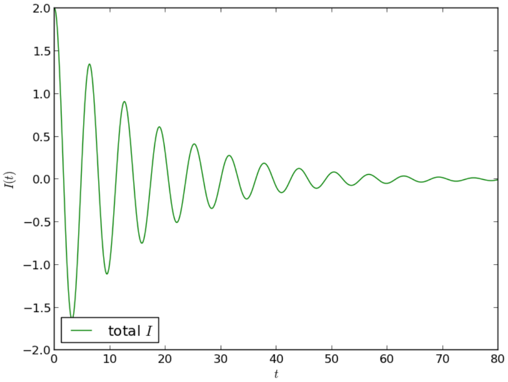
LCR-circuit example: DC case

Figure : Current in the LCR circuit $L = 1$, $C = 1$, $R = 0.125$, $E = 0.25$, with the initial condition $I(0) = 2$ and $I'(0) = 0$



Outline

LCR-circuit example: AC case

- Assume the voltage source is alternating with angular frequency $\omega = 1$
- We have that $C = 1$, $L = 1$, and $R = 0.125$.
- In this case, it's the RMS voltage that is 0.25 volts
- This means that $E(t) = 0.25\sqrt{2}\sin(t)$, and $E'(0) = 0.25\sqrt{2}\cos(t)$, so that I is a solution to the initial value problem

$$I'' + 0.125I' + I = 0.25\sqrt{2}\cos(t)$$

LCR-circuit example: AC case

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LCR-circuit example: AC case

- As per our previous discussion, we write

$$I = I_{\text{transient}} + I_{\text{stable}},$$

- where $I_{\text{transient}}$ is a solution of the homogeneous equation

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- and where I_{stable} is a particular solution of

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LCR-circuit example: AC case

- Using the equation of the previous slide,

$$I_{\text{steady}} = r\cos(\omega t - \delta),$$

- where $\omega_0 = 1$, $\Delta = \sqrt{0.125} = \sqrt{2}/4$, $\delta = \pi/2$, and $r = 1$, so that

$$I_{\text{steady}} = \cos\left(t - \frac{\pi}{2}\right)$$

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LCR-circuit example: AC case

- Also, since $I_{\text{transient}}$ is a solution of the homogeneous equation, we know that

$$I_{\text{transient}} = r_0 e^{-t/16} \cos\left(\frac{\sqrt{255}}{16} t - \delta_0\right),$$

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- EXERCISE: determine r_0 and δ_0

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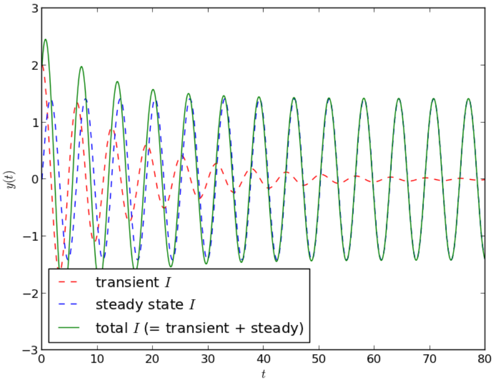

Figure : Current in the LCR circuit $L = 1$, $C = 1$, $R = 0.125$, $E = 0.25$, with the initial condition $I(0) = 2$ and $I'(0) = 0$

Review!

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Next time:

- Review for second midterm

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