# Math 307 Lecture 18 Laplace Transforms of Discontinuous Functions

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June 5, 2014

W.R. Casper Math 307 Lecture 18

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# Today!

### Last time:

- Step Functions
- This time:
  - Laplace Transforms of Discontinuous Functions
  - Differential Equations with Discontinuous Forcing

Next time:

More on Discontinuous Forcing

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- Step Functions and Hat Functions
- Converting From Bracket Form to Step Function Form

2 Laplace Transforms of Piecewise Continuous Functions

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  - A Discontinuous Forcing Example

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Laplace Transforms of Piecewise Continuous Functions Differential Equations with Discontinuous Forcing Step Functions and Hat Functions Converting From Bracket Form to Step Function Form

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# **Step Functions and Hat Functions**

• Last time we defined a step function to be a function of the form

$$u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \ge c \end{cases}$$

$$h_{a,b}(t) = u_a(t) - u_b(t) = \begin{cases} 0, & \text{if } t < a \\ 1, & \text{if } a \le t < b \\ 0, & \text{if } t > b \end{cases}$$

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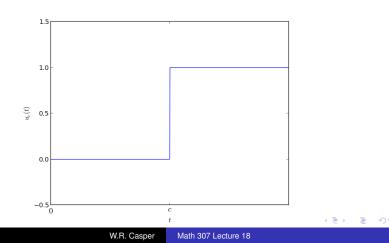
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### Plot of a Step Function

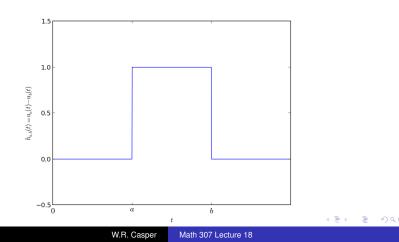
### Figure : Plot of step function $u_c(t)$



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### Plot of a Hat Function

### Figure : Plot of hat function $h_{a,b}(t)$



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# Brackets to Step Functions Example

• Last time we also learned how to convert from a function defined this way:

$$f(t) = \begin{cases} \sin(t) & \text{if } 0 \le t < \pi/4\\ \sin(t) + \cos(t - \pi/4) & \text{if } t \ge \pi/4 \end{cases}$$

$$f(t) = \sin(t)u_0(t) + \cos(t - \pi/4)u_{\pi/4}(t)$$

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# Try it Yourself!

Step Functions and Hat Functions Converting From Bracket Form to Step Function Form

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# Convert the following functions from bracket form to step function form:

 $f(t) = \begin{cases} 1 & \text{if } 0 \le t < 2\\ e^{-(t-2)} & \text{if } t \ge 2 \end{cases}$  $f(t) = \begin{cases} t & \text{if } 0 \le t < 1\\ t-1 & \text{if } 1 \le t < 2\\ t-2 & \text{if } 2 \le t < 3 \end{cases}$ 

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Convert the following functions from bracket form to step function form:

 $f(t) = \begin{cases} t & \text{if } 0 \le t < 1\\ t - 1 & \text{if } 1 \le t < 2\\ t - 2 & \text{if } 2 \le t < 3\\ 0 & \text{if } t \ge 3 \end{cases}$ 

 $f(t) = \begin{cases} 1 & \text{if } 0 \le t < 2\\ e^{-(t-2)} & \text{if } t > 2 \end{cases}$ 

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# Laplace Transform of $f(t - c)u_c(t)$

- Suppose that f(t) is a piecewise continuous functions of exponential type, and that c > 0. Then
- We wish to calculate the Laplace transform of  $f(t c)u_c(t)$
- Computation first shows

$$\mathcal{L}\left\{f(t-c)u_{c}(t)\right\} = \int_{0}^{\infty} e^{-st}f(t-c)u_{c}(t)dt$$
$$= \int_{c}^{\infty} e^{-st}f(t-c)dt$$

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• To summarize

$$\mathcal{L}\left\{f(t-c)u_{c}(t)\right\}=e^{-sc}\mathcal{L}\left\{f(t)\right\}.$$

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- Let F(s) be the Laplace transform of f(t)
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### Summary of Additional Laplace Transform Properties

Additional Laplace transform properties:

 $\mathcal{L}\left\{f(t-c)u_{c}(t)\right\}=e^{-sc}\mathcal{L}\left\{f(t)\right\}$ 

$$\mathcal{L}^{-1}\left\{e^{-sc}F(s)\right\} = f(t-c)u_c(t)$$

$$\mathcal{L}\left\{f(t)e^{ct}\right\}=F(s-c)$$

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### Summary of Additional Laplace Transform Properties

### Additional Laplace transform properties:

 $\mathcal{L}\left\{f(t-c)u_{c}(t)\right\}=e^{-sc}\mathcal{L}\left\{f(t)\right\}$ 

$$\mathcal{L}^{-1}\left\{e^{-sc}F(s)\right\} = f(t-c)u_c(t)$$

$$\mathcal{L}\left\{f(t)e^{ct}\right\}=F(s-c)$$

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Laplace Transforms of Step Functions Transforms of Piecewise Continuous Functions Try it Yourself!

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### Summary of Additional Laplace Transform Properties

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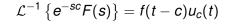
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## Outline

Laplace Transforms of Step Functions Transforms of Piecewise Continuous Functions Try it Yourself!

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### Review of Last Time

- Step Functions and Hat Functions
- Converting From Bracket Form to Step Function Form
- Laplace Transforms of Piecewise Continuous Functions
   Laplace Transforms of Step Functions
  - Transforms of Piecewise Continuous Functions
  - Try it Yourself!
- Differential Equations with Discontinuous Forcing
   A Discontinuous Forcing Example

## A First Example

Laplace Transforms of Step Functions Transforms of Piecewise Continuous Functions Try it Yourself!

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#### Example

Find the Laplace transform of

$$f(t) = \begin{cases} \sin(t) & \text{if } 0 \le t < \pi/4\\ \sin(t) + \cos(t - \pi/4) & \text{if } t \ge \pi/4 \end{cases}$$

• As we saw earlier, we may write

$$f(t) = \sin(t)u_0(t) + \cos(t - \pi/4)u_{\pi/4}(t)$$

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Laplace Transforms of Step Functions Transforms of Piecewise Continuous Functions Try it Yourself!

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Laplace Transforms of Step Functions Transforms of Piecewise Continuous Functions Try it Yourself!

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## A First Example

Laplace Transforms of Step Functions Transforms of Piecewise Continuous Functions Try it Yourself!

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- So what will it's Laplace transform look like?
- Using the Laplace transform of step functions, we see that its Laplace transform is

$$\mathcal{L} \{ f(t) \} = \mathcal{L} \{ \sin(t)u_0(t) \} + \mathcal{L} \{ \cos(t - \pi/4)u_{\pi/4}(t) \}$$
  
=  $e^{-0s}\mathcal{L} \{ \sin(t) \} + e^{-\pi s/4}\mathcal{L} \{ \cos(t) \}$   
=  $\frac{1}{s^2 + 1} + e^{-\pi s/4} \frac{s}{s^2 + 1}$ 

## A First Example

Laplace Transforms of Step Functions Transforms of Piecewise Continuous Functions Try it Yourself!

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Laplace Transforms of Step Functions Transforms of Piecewise Continuous Functions Try it Yourself!

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Laplace Transforms of Step Functions Transforms of Piecewise Continuous Functions Try it Yourself!

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### A Second Example

#### Example

Find the inverse Laplace transform of

$$F(s) = \frac{1 - e^{-2s}}{s^2}$$

We first write

$$F(s) = \frac{1}{s^2} - e^{-2s} \frac{1}{s^2}$$

Laplace Transforms of Step Functions Transforms of Piecewise Continuous Functions Try it Yourself!

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Laplace Transforms of Step Functions Transforms of Piecewise Continuous Functions Try it Yourself!

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### A Second Example

### • Therefore we calculate

$$f(t) = \mathcal{L}^{-1} \{ F(s) \} = \mathcal{L}^{-1} \left\{ \frac{1}{s^2} \right\} - \mathcal{L}^{-1} \left\{ e^{-2s} \frac{1}{s^2} \right\}$$
$$= t - u_2(t)(t - 2)$$

We can but this back into the bracket notation as

$$f(t) = \begin{cases} t & \text{if } t < 2\\ 2 & \text{if } t \ge 2 \end{cases}$$

Laplace Transforms of Step Functions Transforms of Piecewise Continuous Functions Try it Yourself!

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Laplace Transforms of Step Functions Transforms of Piecewise Continuous Functions Try it Yourself!

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## A Third Example

#### Example

Find the inverse Laplace transform of

$$\mathsf{F}(s) = \frac{1}{s^2 - 4s + 5}$$

We first write

$$F(s) = \frac{1}{(s-2)^2 + 1}$$

- Therefore we can write F(s) = G(s-2) for  $G(s) = \frac{1}{s^2+1}$
- Since  $\mathcal{L}^{-1} \{ G(s) \} = \sin(t)$ , it follows that

$$\mathcal{L}^{-1} \{ F(s) \} = \mathcal{L}^{-1} \{ G(s-2) \} = e^{2t} \sin(t)$$

Laplace Transforms of Step Functions Transforms of Piecewise Continuous Functions Try it Yourself!

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# Jous Functions Transforms of Piecewise Try it Yourself!

## Outline

### Review of Last Time

- Step Functions and Hat Functions
- Converting From Bracket Form to Step Function Form

### 2 Laplace Transforms of Piecewise Continuous Functions

- Laplace Transforms of Step Functions
- Transforms of Piecewise Continuous Functions
- Try it Yourself!
- Differential Equations with Discontinuous Forcing
   A Discontinuous Forcing Example

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Laplace Transforms of Step Functions Transforms of Piecewise Continuous Functions Try it Yourself!

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### Laplace Transform Examples

### Find the Laplace Transforms of the following functions

 $f(t) = \begin{cases} 0 & \text{if } t < 2\\ (t-2)^2 & \text{if } t \ge 2 \end{cases}$ 

$$f(t) = \begin{cases} 0 & \text{if } t < \pi \\ t - \pi & \text{if } \pi \le t < 2\pi \\ 0 & \text{if } t \ge 2\pi \end{cases}$$

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Laplace Transforms of Step Functions Transforms of Piecewise Continuous Functions Try it Yourself!

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#### Inverse Laplace Transform Examples

$$F(s) = \frac{3!}{(s-2)^4}$$

$$F(s) = \frac{2(s-1)e^{-2s}}{s^2 - 2s + 2}$$

$$F(s) = \frac{(s-2)e^{-s}}{s^2 - 4s + 3}$$

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   A Discontinuous Forcing Example

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## An Example

#### Example

#### Find the solution of the initial value problem

$$2y'' + y' + 2y = g(t), \ y(0) = 0, \ y'(0) = 0$$

A Discontinuous Forcing Example

where

$$g(t) = u_5(t) - u_{20}(t) = \left\{ egin{array}{cc} 0 & ext{if } 0 \leq t < 5 \ 1 & ext{if } 5 \leq t < 20 \ 0 & ext{if } t \geq 20 \end{array} 
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A Discontinuous Forcing Example

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# An Example

- This IVP models the charge on a capacitor in an LCR circuit where a battery is connected at t = 5 and disconnected at t = 20
- We calculate using y(0) = 0 and y'(0) = 0

$$\mathcal{L}\left\{y'\right\} = s\mathcal{L}\left\{y\right\} - y(0) = s\mathcal{L}\left\{y\right\}$$

A Discontinuous Forcing Example

and

$$\mathcal{L}\left\{y''\right\} = s^{2}\mathcal{L}\left\{y\right\} - sy(0) - y'(0) = s^{2}\mathcal{L}\left\{y\right\}.$$

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A Discontinuous Forcing Example

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A Discontinuous Forcing Example

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# An Example

#### Moreover

$$\mathcal{L}\left\{g(t)\right\} = \mathcal{L}\left\{u_{5}(t)\right\} - \mathcal{L}\left\{u_{2}0\right\}(t) = e^{-5s}\frac{1}{s} - e^{-20s}\frac{1}{s}$$

 so taking the Laplace transform of both sides of the original differential equation

$$2y'' + y' + 2y = g(t)$$

gives us

$$2s^{2}\mathcal{L}\{y\} + s\mathcal{L}\{y\} + 2\mathcal{L}\{y\} = \frac{e^{-5s} - e^{-20s}}{s}.$$

A Discontinuous Forcing Example

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# An Example

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A Discontinuous Forcing Example

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A Discontinuous Forcing Example

 so taking the Laplace transform of both sides of the original differential equation

$$2y''+y'+2y=g(t)$$

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$$2s^{2}\mathcal{L}\{y\} + s\mathcal{L}\{y\} + 2\mathcal{L}\{y\} = \frac{e^{-5s} - e^{-20s}}{s}.$$

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# An Example

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A Discontinuous Forcing Example

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# An Example

#### • After a little algebra, this tells us

$$\mathcal{L}\left\{y\right\} = \frac{e^{-5s} - e^{-20s}}{s(2s^2 + s + 2)} = (e^{-5s} - e^{-20s})H(s)$$

A Discontinuous Forcing Example

$$y = \mathcal{L}^{-1} \left\{ (e^{-5s} - e^{-20s}) H(s) \right\}$$
  
=  $\mathcal{L}^{-1} \left\{ e^{-5s} H(s) \right\} - \mathcal{L}^{-1} \left\{ e^{-20s} H(s) \right\}$   
=  $h(t - 5) u_5(t) - h(t - 20) u_{20}(t)$ 

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#### A Discontinuous Forcing Example

### An Example

#### • After a little algebra, this tells us

$$\mathcal{L}\left\{y\right\} = \frac{e^{-5s} - e^{-20s}}{s(2s^2 + s + 2)} = (e^{-5s} - e^{-20s})H(s)$$

• for 
$$H(s) = 1/(s(2s^2 + s + 2))$$
  
• Thus if  $h(t) = \mathcal{L}^{-1} \{H(s)\}$ , then

$$y = \mathcal{L}^{-1} \left\{ (e^{-5s} - e^{-20s}) H(s) \right\}$$
  
=  $\mathcal{L}^{-1} \left\{ e^{-5s} H(s) \right\} - \mathcal{L}^{-1} \left\{ e^{-20s} H(s) \right\}$   
=  $h(t - 5) u_5(t) - h(t - 20) u_{20}(t)$ 

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#### A Discontinuous Forcing Example

#### An Example

#### • After a little algebra, this tells us

$$\mathcal{L}\left\{y\right\} = \frac{e^{-5s} - e^{-20s}}{s(2s^2 + s + 2)} = (e^{-5s} - e^{-20s})H(s)$$

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A Discontinuous Forcing Example

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# An Example

#### • Now using partial fractions

$$\frac{1}{s(2s^2+s+2)} = \frac{a}{s} + \frac{bs+c}{2s^2+s+2},$$

• one easily determines that a = 1/2, b = 1 and c = 1/2, so

$$H(s) = \frac{1/2}{s} - \frac{s + \frac{1}{2}}{2s^2 + s + 2}$$

A Discontinuous Forcing Example

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A Discontinuous Forcing Example

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A Discontinuous Forcing Example

# An Example

- The inverse Laplace transform of  $\frac{1/2}{s}$  is easy
- The inverse Laplace transform of  $\frac{s+\frac{1}{2}}{2s^2+s+2}$  is more difficult

A Discontinuous Forcing Example

- How do we find it?
- Complete the square in the denominator

$$\frac{s+\frac{1}{2}}{2s^2+s+2} = \frac{s+\frac{1}{2}}{2(s+\frac{1}{4})^2+\frac{15}{8}}$$

• Factor out a 1/2

$$\frac{s+\frac{1}{2}}{2s^2+s+2} = \frac{1}{2} \frac{s+\frac{1}{2}}{(s+\frac{1}{4})^2 + \frac{15}{16}}$$

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## An Example

• Lastly, try to put this in a form that looks like a linear combination of translations of Laplace transforms of  $sin(\sqrt{15}t/4)$  and  $cos(\sqrt{15}t/4)$ 

$$\frac{s+\frac{1}{2}}{2s^2+s+2} = \frac{1}{2} \frac{(s+\frac{1}{4})+\frac{1}{4}}{(s+\frac{1}{4})^2+\frac{15}{16}}$$
$$= \frac{1}{2} \frac{(s+\frac{1}{4})}{(s+\frac{1}{4})^2+\frac{15}{16}} + \frac{1}{2} \frac{\frac{1}{4}}{(s+\frac{1}{4})^2+\frac{15}{16}}$$
$$= \frac{1}{2} \frac{(s+\frac{1}{4})}{(s+\frac{1}{4})^2+\frac{15}{16}} + \frac{1}{2\sqrt{15}} \frac{\frac{\sqrt{15}}{4}}{(s+\frac{1}{4})^2+\frac{15}{16}}$$

A Discontinuous Forcing Example

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A Discontinuous Forcing Example

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A Discontinuous Forcing Example

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# An Example

A Discontinuous Forcing Example

#### • Therefore

$$\frac{s + \frac{1}{2}}{2s^2 + s + 2} = \frac{1}{2}\mathcal{L}\left\{e^{-t/4}\cos(\sqrt{15}t/4)\right\} + \frac{1}{2\sqrt{15}}\mathcal{L}\left\{e^{-t/4}\sin(\sqrt{15}t/4)\right\}$$

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A Discontinuous Forcing Example

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A Discontinuous Forcing Example

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# An Example

#### • Thus we have shown that

$$y = h(t-5)u_5(t) - h(t-20)u_{20}(t)$$

A Discontinuous Forcing Example

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$$h(t) = \frac{1}{2} - \frac{1}{2}e^{-t/4}\cos(\sqrt{15}t/4) - \frac{1}{2\sqrt{15}}e^{-t/4}\sin(\sqrt{15}t/4)$$

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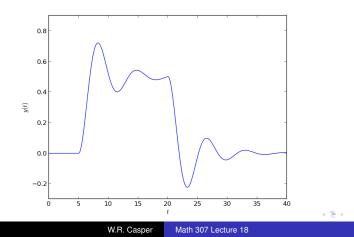
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A Discontinuous Forcing Example

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#### Plot of a Solution to IVP

Figure : Plot of Solution to IVP with Discontinuous Forcing



### **Review!**

A Discontinuous Forcing Example

#### Today:

• Fun with Laplace Transforms!

Next time:

• More fun with Laplace Transforms!

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