Math 307 Quiz 7

June 4, 2014

Problem 1. For each of the following, determine the correct form of the partial fraction decomposition

Example The form of the PFD of $\frac{3s+4}{(s-1)^2}$ is $\frac{A}{s-1} + \frac{B}{(s-1)^2}$

(a) $\frac{2s+3}{s^2+2s+1}$

(b) $\frac{s^3+3s^2+4s+5}{(s-2)^2(s-3)}$

(c) $\frac{s^2+17s+43}{s(s^2+1)^2(s-4)^2(s-5)}$

(d) $\frac{s^2+3s+2}{(s-1)(s+4)}$

Solution 1.

(a)

$$\frac{A}{s+1} + \frac{B}{(s+1)^2}$$

(b) By polynomial long division:

$$\frac{s^3+3s^2+4s+5}{(s-2)^2(s-3)} = 1 + \frac{10s^2-12s+17}{(s-2)^2(s-3)}$$

and therefore the PFD is of the form

$$1 + \frac{A}{(s-2)} + \frac{B}{(s-2)^2} + \frac{C}{(s-3)}$$

(c)

$$\frac{A}{s} + \frac{Bs + C}{(s^2+1)} + \frac{Ds + E}{(s^2+1)^2} + \frac{F}{s-4} + \frac{G}{(s-4)^2} + \frac{H}{(s-5)}$$

(d) By polynomial long division:

$$\frac{s^2 + 3s + 2}{(s-1)(s+4)} = 1 + \frac{6}{(s-1)(s+4)}$$

and therefore the PFD is of the form

$$1 + \frac{A}{s-1} + \frac{B}{s+4}$$

Problem 2.

(a) State the definition of the Laplace trasform $F(s) = \mathcal{L}(f(t))$.

(b) Give an example of a function whose Laplace transform does not exist, and explain why.

(c) Determine the Laplace transform $F(s)$ of

$$f(t) = te^{2t}\cos(t)$$

using only the basic definition of the Laplace transform.

Solution 2.

(a)

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t) dt.$$

(b) $f(t) = e^{t^2}$, since it is not of exponential type

(c) Notice that

$$f(t) = \mathrm{Re}(te^{(2+i)t}),$$

and therefore

$$F(s) = \mathcal{L}(\mathrm{Re}(te^{(2+i)t})) = \mathrm{Re}\mathcal{L}(te^{(2+i)t}).$$

Then we calculate (for $s \geq 2$)

$$\begin{aligned}
\mathcal{L}(te^{(2+i)t}) &= \int_0^\infty te^{(2+i)t}e^{-st}dt = \int_0^\infty te^{(2-s+i)t}dt \\
&= \frac{t}{2-s+i}e^{(2-s+i)t}\Big|_0^\infty - \int_0^\infty \frac{1}{2-s+i}e^{(2-s+i)t}dt \\
&= 0 - \int_0^\infty \frac{1}{2-s+i}e^{(2-s+i)t}dt = -\frac{1}{(2-s+i)^2}e^{(2-s+i)t}\Big|_0^\infty \\
&= \frac{1}{(2-s+i)^2} = \frac{(s-2)^2-1}{((s-2)^2+1)^2} + i\frac{2(s-2)}{((s-2)^2+1)^2}
\end{aligned}$$

and therefore

$$F(s) = \frac{(s-2)^2-1}{((s-2)^2+1)^2}.$$

Problem 3. Determine the inverse Laplace transform of

$$F(s) = \frac{2s+3}{s^2+4s+1}$$

Solution 3. First notice that s^2+4s+1 has roots $-2 \pm \sqrt{3}$ and therefore factors as

$$s^2+4s+1 = (s+2-\sqrt{3})(s+2+\sqrt{3}).$$

Consequently,

$$F(s) = \frac{2s+3}{(s+2-\sqrt{3})(s+2+\sqrt{3})}$$

so F has the PFD form:

$$\frac{2s+3}{(s+2-\sqrt{3})(s+2+\sqrt{3})} = \frac{A}{s+2-\sqrt{3}} + \frac{B}{s+2+\sqrt{3}}$$

solving this, we find $A = 1 - \frac{1}{2\sqrt{3}}$ and $B = 1 + \frac{1}{2\sqrt{3}}$ so that

$$F(s) = \left(1 - \frac{1}{2\sqrt{3}}\right)\frac{1}{s+2-\sqrt{3}} + \left(1 + \frac{1}{2\sqrt{3}}\right)\frac{1}{s+2+\sqrt{3}}$$

and therefore

$$f(t) = \left(1 - \frac{1}{2\sqrt{3}}\right)e^{(-2+\sqrt{3})t} + \left(1 + \frac{1}{2\sqrt{3}}\right)e^{(-2-\sqrt{3})t}$$

Problem 4. Determine the inverse Laplace transform of

$$F(s) = \frac{s-5}{s^2-s-6}$$

We again factor

$$F(s) = \frac{s-5}{(s-3)(s+2)}$$

so $F(s)$ has the PFD:

$$\frac{s-5}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{(s+2)}.$$

Solving for A and B, we obtain $A = 8/5$ and $B = -3/5$, so that

$$F(s) = \frac{8}{5}\frac{1}{s-3} + -\frac{3}{5}\frac{1}{s+2}.$$

Thus

$$f(t) = \frac{8}{5}e^{3t} - \frac{3}{5}e^{-2t}.$$