Math 307 Quiz 7

June 6, 2014

Problem 1. For each of the following, determine the correct form of the partial fraction decomposition

Example The form of the PFD of $\frac{3s+4}{(s-1)^2}$ is $\frac{A}{s-1} + \frac{B}{(s-1)^2}$

$$\text{(a)}\ \frac{2s+3}{s^2+2s+1} \quad \text{(b)}\ \frac{3s^2}{(s^2+s+4)(s-2)} \quad \text{(c)}\ \frac{3s+5}{(s^2+4s+3)}$$

Solution 1. (a) $\frac{A}{s+1} + \frac{B}{(s+1)^2}$

(b) $\frac{A}{s-1} + \frac{Bs+C}{s^2+s+4}$

(c) $\frac{A}{s+3} + \frac{B}{s+1}$

Problem 2.

(a) State the definition of the Laplace trasform $F(s) = \mathcal{L}(f(t))$.

(b) Determine the Laplace transform $F(s)$ of

$$f(t) = e^{2t}\sin(t)$$

using only the basic definition of the Laplace transform.

Solution 2. 1. $F(s) = \mathcal{L}(f(t)) = \int_0^\infty f(t)e^{-st}dt$

2. First notice that $f(t) = \mathrm{Im}(e^{(2+i)t})$ and therefore

$$\begin{aligned} F(s) &= \mathcal{L}(\mathrm{Im}(e^{(2+i)t})) = \mathrm{Im}\mathcal{L}(e^{(2+i)t}) \\ &= \mathrm{Im}\int_0^\infty e^{(2-s+i)t}dt = \mathrm{Im}\frac{1}{s-2-i} \\ &= \mathrm{Im}\left(\frac{s-2}{(s-2)^2+1} + i\frac{1}{(s-2)^2+1}\right) = \frac{1}{(s-2)^2+1}. \end{aligned}$$

Problem 3. Determine the inverse Laplace transform of

$$F(s) = \frac{3s+5}{s^2+s+4}$$

Solution 3. We complete the square in the denominator to write

$$F(s) = \frac{3s+5}{(s+1/2)^2+15/4}.$$

We wish to write this as a linear combination

$$F(s) = A\frac{s+1/2}{(s+1/2)^2+15/4} + B\frac{1}{(s+1/2)^2+15/4}$$

for some constants A, B. This means that

$$\frac{3s+5}{(s+1/2)^2+15/4} = A\frac{s+1/2}{(s+1/2)^2+15/4} + B\frac{\sqrt{15}/2}{(s+1/2)^2+15/4}$$

and clearing denominators, we get

$$3s+5 = A(s+1/2) + B\sqrt{15}/2$$

from which we immediately see that $A = 3$ and $B = 7/\sqrt{15}$. Thus at long last:

$$F(s) = 3\frac{s+1/2}{(s+1/2)^2+15/4} + \frac{7}{\sqrt{15}}\frac{\sqrt{15}/2}{(s+1/2)^2+15/4}$$

and therefore

$$f(t) = 3e^{-t/2}\cos(\sqrt{15}t/2) + \frac{7}{\sqrt{15}}\sin(\sqrt{15}t/2)$$

Problem 4. Determine the inverse Laplace transform of

$$F(s) = \frac{s-5}{s^2-s-6}$$

Solution 4. We perform a PFD:

$$F(s) = \frac{s-5}{s^2-s-6} = \frac{A}{s-3} + \frac{B}{s+2}$$

and clearing denominators, we find

$$A(s+2) + B(s-3) = s - 5.$$

Plugging in $s = -2$, we get $B(-5) = -7$ so that $B = 7/5$. Plugging in $s = 3$, we get $A(5) = -2$ so that $A = -2/5$, and thus

$$F(s) = \frac{-2/5}{s-3} + \frac{7/5}{s+2}.$$

Therefore

$$f(t) = -\frac{2}{5}e^{3t} + \frac{7}{5}e^{-2t}.$$