# Math 307 Quiz 8

June 17, 2014

**Problem 1.** Find the inverse Laplace transform of

$$F(s) = \frac{s+3}{s^2+s+6}$$

**Solution 1.** We complete the square, writing

$$F(s) = \frac{s+3}{(s+1/2)^2+23/4}.$$

We wish to write $F$ as a linear combination

$$F(s) = A\frac{s+1/2}{(s+1/2)^2+23/4} + B\frac{\sqrt{23}/2}{(s+1/2)^2+23/4}.$$

This means that

$$A\frac{s+1/2}{(s+1/2)^2+23/4} + B\frac{\sqrt{23}/2}{(s+1/2)^2+23/4} = \frac{s+3}{(s+1/2)^2+23/4}$$

and clearing denominators, we get

$$A(s+1/2) + B\sqrt{23}/2 = s+3,$$

and therefore $A = 1$ and $B = 5/\sqrt{23}$. Thus

$$F(s) = \frac{s+1/2}{(s+1/2)^2+23/4} + \frac{5}{\sqrt{23}}\frac{\sqrt{23}/2}{(s+1/2)^2+23/4}$$

and therefore

$$f(t) = e^{-t/2}\cos(\sqrt{23}t/2) + \frac{5}{\sqrt{23}}e^{-t/2}\sin(\sqrt{23}t/2).$$

**Problem 2.** Consider the function

$$f(t) = \begin{cases} 0, & t < \pi \\ 1, & \pi \le t < 5 \\ 0, & 5 \le t \end{cases}$$

(a) Express $f(t)$ in step function form

(b) Find the Laplace transform of $f(t)$

**Solution 2.**

(a) $f(t) = u_\pi(t) - u_5(t)$

(b) $F(s) = e^{-\pi s}/s - e^{-5s}/s$

**Problem 3.** Use Laplace transforms to find the solution to the differential equation

$$y'' - 9y = e^t \sin(2t),\ y(0) = 1,\ y'(0) = 2.$$

**Solution 3.** We first calculate

$$\mathcal{L}(y'') = s^2\mathcal{L}(y) - sy(0) - y'(0) = s^2\mathcal{L}(y) - s - 2$$

and also

$$\mathcal{L}(e^t \sin(2t)) = \frac{2}{(s-1)^2+4},$$

so that taking the Laplace transform of both sides of the original differential equation gives us

$$s^2\mathcal{L}(y) - s - 2 - 9\mathcal{L}(y) = \frac{2}{(s-1)^2+4}.$$

It follows that

$$\mathcal{L}(y) = \frac{s+2}{s^2-9} + \frac{2}{((s-1)^2+4)(s^2-9)}.$$

Now by partial fractions, we calculate

$$\frac{2}{((s-1)^2+4)(s^2-9)} = \frac{(-1/40)s - 1/8}{(s-1)^2+4} + \frac{1/24}{s-3} - \frac{1/60}{s+3}.$$

And also

$$\frac{s+2}{s^2-9} = \frac{5/6}{s-3} + \frac{1/6}{s+3}$$

so that

$$\mathcal{L}(y) = \frac{(-1/40)s - 1/8}{(s-1)^2+4} + \frac{21/24}{s-3} - \frac{11/60}{s+3}.$$

Lastly, notice that

$$\frac{(-1/40)s - 1/8}{(s-1)^2+4} = -\frac{1}{40}\frac{s-1}{(s-1)^2+4} - \frac{3}{40}\frac{2}{(s-1)^2+4}$$

and therefore

$$\mathcal{L}^{-1}\left(\frac{(-1/40)s - 1/8}{(s-1)^2+4}\right) = -\frac{1}{40}e^t\cos(2t) - \frac{3}{40}e^t\sin(2t)$$

and

$$\mathcal{L}^{-1}\left(\frac{1/24}{s-3}\right) = \frac{1}{24}e^{3t}$$

and

$$\mathcal{L}^{-1}\left(\frac{1/60}{s+3}\right) = \frac{1}{60}e^{-3t},$$

so that

$$y = -\frac{1}{40}e^t\cos(2t) - \frac{3}{40}e^t\sin(2t) + \frac{1}{24}e^{3t} - \frac{1}{60}e^{-3t}.$$