Math 307 Section M
Spring 2015
Midterm 1
April 22, 2015
Time Limit: 50 Minutes

Name (Print): ____________________

Student ID: ____________________

This exam contains 12 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam. However, you may use a single, handwritten, one-sided notesheet and a *basic* calculator.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- **Box Your Answer** where appropriate, in order to clearly indicate what you consider the answer to the question to be.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Do not write in the table to the right.

1. Solve the following initial value problems

   (a) (5 points) Solve the following initial value problems:

$$\frac{dy}{dx} = xe^x/y^2 + x/y^2, \quad y(0) = 1$$

   (b) (5 points)

$$y' = \frac{x+y}{x-y}, \quad y(1) = 1$$

Solution 1.

(a) The equation is separable! We separate:

$$y^2 dy = (xe^x + x)dx$$

and integrate:

$$\frac{1}{3}y^3 = xe^x - e^x + \frac{1}{2}x^2 + C.$$

Solving for y, we the obtain

$$y = \left(3xe^x - 3e^x + \frac{3}{2}x^2 + C\right)^{1/3}.$$

Lastly we use the initial condition to find $C = 4$. Therfore the solution is

$$y = \left(3xe^x - 3e^x + \frac{3}{2}x^2 + 4\right)^{1/3}.$$

(b) The equation is homogeneous! To see this, merely note that the right hand side simplifies to

$$\frac{x+y}{x-y} = \frac{x+y}{x-y}\frac{1/x}{1/x} = \frac{1+y/x}{1-y/x}$$

We use the substitution $z = y/x$ and $y' = z + xz'$ to find

$$z + xz' = \frac{1+z}{1-z}.$$

Therefore we obtain the separable equation

$$xz' = \frac{1+z^2}{1-z}$$

Separating and integrating, we obtain

$$\int \frac{1-z}{1+z^2}dz = \int \frac{1}{x}dx.$$

How do we do the integral on the left hand side? We can write

$$\int \frac{1-z}{1+z^2}dz = \int \frac{1}{1+z^2}dz - \int \frac{z}{1+z^2}dz = \tan^{-1}(z) - \frac{1}{2}\ln(1+z^2).$$

Therefore we have

$$\tan^{-1}(z) - \frac{1}{2}\ln(1+z^2) = \ln|x| + C$$

The initial condition $y(1) = 1$ tells us that $z(1) = 1$, and therefore $C = \pi/4 - \ln(2)/2$. Thus

$$\tan^{-1}(y/x) - \frac{1}{2}\ln(1+(y/x)^2) = \ln|x| + \frac{\pi}{4} - \ln(2)/2.$$

In this situation, it is too complicated to solve for y in terms of x.

2.

(a) (5 points) Find the general solution to the linear equation

$$y' = 3y + \cos(2x)$$

using the method of integrating factors.

(b) (5 points) Find the general solution to the linear equation

$$y' = -\cot(x)y + \cos(x) + 3$$

using the method of variation of parameters.

Solution 2.

(a) We have an integrating factor given by the equation $\mu(x) = e^{\int -3dx} = e^{-3x}$. Multiplying by $\mu(x)$ we obtain the equation

$$e^{-3x}y' - 3e^{3x}y = e^{-3x}\cos(2x).$$

The left hand side is equal to $(e^{-3x}y)'$, and therefore

$$(e^{-3x}y)' = e^{-3x}\cos(2x).$$

Integrating both sides with respect to x, we thus obtain

$$e^{-3x}y = \frac{-3}{13}e^{-3x}\cos(2x) + \frac{2}{13}e^{-3x}\sin(2x) + C$$

Therefore

$$y = \frac{-3}{13}e^{-3x} + \frac{2}{13}\sin(2x) + Ce^{3x}.$$

(b) To use the method of variation of parameters, we first solve the corresponding homogeneous equation

$$y_h' = -\cot(x)y_h.$$

This equation is separable, separating and integrating, we obtain

$$\int 1y_h dy_h = -\int \cot(x)dx.$$

Therefore

$$\ln(y_h) = -\ln(\sin(x)) + C,$$

and we can take our constant C to be 0, obtaining the homogeneous solution $y_h = \csc(x)$. Then we know from the method of variation of parameters that $y = y_h v$ for

$$v = \int \frac{q}{y_h}dx = \int \sin(x)\cos(x) + 3\sin(x)dx = \frac{1}{2}\sin^2(x) - 3\cos(x) + C.$$

Thus

$$y = y_h v = \frac{1}{2}\sin(x) - 3\cot(x) + C\csc(x).$$

3. (a) (2 points) Give an example of an initial value problem with no solution

(b) (2 points) Give an example of an initial value problem with more than one solution

(c) (2 points) Without solving the equation, on what interval can we expect a unique solution to the initial value problem

$$\sin(x)y' = \cos(x)y + \frac{1}{1-x}, \quad y(\pi/2) = 1.$$

(d) (2 points) Write the complex number $e^{i\pi/6}$ in $a + ib$ form

(e) (2 points) Write the complex number $\frac{2+3i}{3-2i}$ in $a + ib$ form

Solution 3.

(a) $y' = 1/x$, $y(0) = 1$

(b) $y' = y^{1/3}$, $y(1) = 0$

(c) $(1, \pi)$

(d) $e^{i\pi/6} = \cos(\pi/6) + i\sin(\pi/6) = \frac{\sqrt{3}}{2} + i\frac{1}{2}$.

(e) $\frac{2+3i}{3-2i} = \frac{2+3i}{3-2i}\frac{3+2i}{3+2i} = \frac{(2+3i)(3+2i)}{13} = i$

4. (a) (5 points) Find an integrating factor for the equation

$$ye^{xy} + xy(y+1)e^{xy} + (x + x^2 y)e^{xy}y'.$$

You do NOT need to solve it.

(b) (5 points) Show that the equation

$$2x + 3y - \frac{2x}{(x^2+y^2)^2} + \left(3x - \frac{2y}{(x^2+y^2)^2} - y\right) y' = 0$$

is exact. Then find a family of solutions.

Solution 4.

(a) We try for an integrating factor of the form $\mu(x,y) = \mu(x)$. Then the equation

$$\overbrace{\mu(x)(ye^{xy} + xy(y+1)e^{xy})}^{M(x,y)} + \overbrace{(x + x^2y)\mu(x)e^{xy}}^{N(x,y)} y' = 0.$$

should be exact. This means that $M_y(x,y) = N_x(x,y)$. We calculate

$$\begin{aligned} M_y(x,y) &= \mu(x)(1 + 2xy + x)e^{xy} + \mu(x)(y + xy^2 + xy)xe^{xy} \\ &= \mu(x)(1 + 3xy + x + x^2y + x^2y^2)e^{xy}. \end{aligned}$$

and also

$$\begin{aligned} N_x(x,y) &= \mu'(x)(x + x^2y)e^{xy} + \mu(x)(1 + 2xy)e^{xy} + \mu(x)(x + x^2y)ye^{xy} \\ &= \mu'(x)(x + x^2y)e^{xy} + \mu(x)(1 + 3xy + x^2y^2)e^{xy}. \end{aligned}$$

Therefore setting $M_y = N_x$ we obtain

$$\mu(x)(1 + 3xy + x + x^2y + x^2y^2)e^{xy} = \mu'(x)(x + x^2y)e^{xy} + \mu(x)(1 + 3xy + x^2y^2)e^{xy}.$$

Simplifying, this becomes

$$\mu(x)(x + x^2y)e^{xy} = \mu'(x)(x + x^2y)e^{xy}.$$

Dividing both sides by $(x + x^2y)e^{xy}$ we then obtain

$$\mu(x) = \mu'(x).$$

Therefore $\mu(x) = e^x$ is an integrating factor.

(b) To show that the equation is exact, we calculate

$$M_y = 3 + \frac{8xy}{(x^2 + y^2)^3} = N_x.$$

Then we have

$$\psi(x,y) = \int M(x,y)\partial x = \int 2x + 3y - \frac{2x}{(x^2 + y^2)^2}\partial x = x^2 + 3xy + \frac{1}{x^2 + y^2} + h(y).$$

In particular, this shows that

$$\psi_y(x,y) = 3x - \frac{2y}{(x^2 + y^2)^2} + h'(y).$$

Then since $\psi_y = N$, we obtain $h'(y) = -y$, and therefore we can take $h(y) = -\frac{1}{2}y^2$. It follows that

$$\psi(x,y) = x^2 + 3xy + \frac{1}{x^2 + y^2} + \frac{1}{2}y^2.$$

Thus a family of solutions is defined implicitly by

$$x^2 + 3xy + \frac{1}{x^2 + y^2} + \frac{1}{2}y^2 = C.$$

5. (10 points) **Rubber Band Ball Drop**

In March 2003, the worlds largest rubber band ball had a mass of roughly 1000 kg and a diameter of 1.4 meters. For science, it was dropped from an airplane at a height of about 1.6 kilometers from the surface. During its descent, the ball experienced two forces: the gravitational force F_g given by

$$F_g = -mg$$

and the force of drag

$$F_d = \frac{1}{2}\rho v^2 C_D \pi r^2.$$

Here, the ρ is the density of air ($\rho = 1.225$ kg/m^3), C_D is the drag coefficient ($C_D = 0.4$), r is the radius, m is the mass and g is the acceleration due to gravity ($g = 9.81$ m/s^2).

(a) Set up an initial value problem describing the vertical velocity of the rubber band ball as a function of time

(b) Find a solution to the differential equation in (a) and use it to determine the *terminal velocity* v_t of the rubber band ball – ie. the velocity of the ball at large times (it appears as a horizontal asymptote of $v(t)$)

(c) In this situation, the velocity of the ball very quickly approaches the terminal velocity, so a good estimation for the velocity of the ball during most of the descent is constant v_t. By approximating that the ball's velocity is always v_t, estimate the time it takes the ball to hit the ground.

Solution 5.

(a) By Newton's law, we know that $F = mv'$. Since $F = F_g + F_d$, this tells us

$$mv' = -mg + \frac{1}{2}\rho v^2 C_D \pi r^2.$$

Furthermore, since the rubber band ball is "dropped", the initial vertical velocity should be 0, so the desired IVP is

$$mv' = -mg + \frac{1}{2}\rho v^2 C_D \pi r^2, \ v(0) = 0.$$

(b) The above equation is separable! However, the number of different symbols is making us dizzy, so let's define a new symbol $\gamma = \frac{1}{2}\rho C_D \pi r^2$ so that the equation that we are trying to solve is simply

$$mv' = -mg + \gamma v^2$$

Separating and integrating, we obtain:

$$\int \frac{1}{-g + (\gamma/m)v^2} dv = \int 1 dt.$$

Therefore since

$$\int \frac{1}{-g + (\gamma/m)v^2} dv = \frac{-m}{\gamma} \int \frac{1}{mg/\gamma - v^2} dv = \frac{-\sqrt{m\gamma}}{2\sqrt{g}} \ln\left(\frac{\sqrt{mg/\gamma} + v}{\sqrt{mg/\gamma} - v}\right) + C$$

Therefore

$$\frac{-\sqrt{m\gamma}}{2\sqrt{g}} \ln\left(\frac{\sqrt{mg/\gamma} + v}{\sqrt{mg/\gamma} - v}\right) = t + C$$

and the initial condition tells us $C = 0$. Let's solve for v. Multiplying by $-2\sqrt{g/m\gamma}$ and exponentiating, we get

$$\frac{\sqrt{mg/\gamma} + v}{\sqrt{mg/\gamma} - v} = -e^{-2\sqrt{g/m\gamma}t}$$

Then after some algebra

$$v = -\sqrt{mg/\gamma}\frac{1 - e^{-2\sqrt{g/m\gamma}t}}{1 + e^{-2\sqrt{g/m\gamma}t}}.$$

Taking $t \to \infty$, we obtain $v_t = -\sqrt{mg/\gamma}$.

(c) We estimate the time t_f when it hits by

$$t_f = 1600/v_t \approx 19 \text{ seconds}.$$