Math 307 Section M
Spring 2015
Exam 2
May 22, 2015
Time Limit: 50 Minutes

Name (Print): ____________________

Student ID: ____________________

This exam contains 9 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam. However, you may use a single, handwritten, one-sided notesheet and a *basic* calculator.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- **Box Your Answer** where appropriate, in order to clearly indicate what you consider the answer to the question to be.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	20	
Total:	70	

1. (10 points) Solve the following initial value problem

$$y'' + 6y' + 9y = 0, \quad y(0) = 1, \; y'(0) = 0$$

Solution 1. The characteristic polynomial is $r^2 + 6r + 9 = (r + 3)^2$. Therefore the general solution looks like

$$y = (Ax + B)e^{-3x}.$$

Since $y(0) = 1$, it follows that $B = 1$, and since $y'(0) = 0$, we must have $A = 3$. Therefore

$$y = (3x + 1)e^{-3x}.$$

2. **Propose a Solution Section!**

Directions: The "Propose a Solution" section consists of five linear nonhomogeneous equations. For each of these equations, write down the type of function y (with undetermined coefficients) you would try, in order to get a particular solution. *You do NOT need to solve the equations* For example, if the equation were

$$y'' + 2y' + y = e^t,$$

a *correct answer* would be

$$y = Ae^t,$$

and *incorrect answers* would include

$$y = (At + B)e^t, \quad y = At^2e^{2t}, \quad y = Ae^{3t}, \quad y = A\pi^t$$

Each part is worth 2pts:

(a) (2 points)
$$4y'' + 5y' + y = t^3e^{2t}$$

(b) (2 points)
$$y'' + y' + 3y = (t + 1)e^{-t}$$

(c) (2 points)
$$y'' + 4y' - 5y = e^t$$

(d) (2 points)
$$y'' + 12y' = t + 17$$

(e) (2 points)
$$y'' - 4y' + 4y = e^{2t}$$

Solution 2.

(a) $y_p = (A + Bt + Ct^2 + Dt^3)e^{2t}$
(b) $y_p = (A + Bt)e^{-t}$
(c) $y_p = Ate^t$
(d) $y_p = At + Bt^2$
(e) $y_p = At^2e^{2t}$

3. (10 points) Find a family of solutions to the differential equation using the method of integrating factors
$$y' = -\frac{3xy + y^2}{x^2 + xy}.$$

Solution 3. One may note that this equation is homogeneous, and therefore that we can actually use $z = y/x$ to find a solution pretty quick. However, the point of this problem is to test your ability to use the method of integrating factors. During the exam, we also gave you the hint to clear the denominator, making the equation:

$$3xy + y^2 + (x^2 + xy)y' = 0.$$

We propose an integrating factor of the form $\mu(x, y) = \mu(x)$. Then

$$\overbrace{3xy\mu(x) + y^2\mu(x)}^{M(x,y)} + \overbrace{(x^2 + xy)\mu(x)}^{N(x,y)}\, y' = 0$$

must be exact. This implies $M_y = N_x$. We calculate

$$M_y(x, y) = (3x + 2y)\mu(x)$$

$$N_y(x, y) = (x^2 + xy)\mu'(x) + (2x + y)\mu(x).$$

Therefore $M_y = N_x$ tells us

$$(3x + 2y)\mu(x) = (2x + y)\mu(x) + (x^2 + xy)\mu'(x).$$

Simplifying, this becomes
$$(x + y)\mu(x) = (x^2 + xy)\mu'(x).$$

Dividing both sides by $(x + y)$, we then get

$$\mu(x) = x\mu'(x).$$

Solving this separable equation, we see that we can take $\mu(x) = x$. Therefore we have the exact equation
$$\overbrace{3x^2y + xy^2}^{M(x,y)} + \overbrace{(x^3 + x^2y)}^{N(x,y)}\, y' = 0$$

Now we need to find $\psi(x, y)$ such that $\psi_x = M$ and $\psi_y = N$. This means that

$$\psi = \int \psi_x(x, y)\partial x = \int M(x, y)\partial x = x^3y + \frac{1}{2}x^2y^2 + h(y).$$

Therefore
$$\psi_y(x, y) = \frac{\partial}{\partial y}(x^3y + \frac{1}{2}x^2y^2 + h(y)) = x^3 + x^2y + h'(y).$$

However, since $\psi_y = N(x, y)$ this means

$$x^3 + x^2y + h'(y) = x^3 + x^2y.$$

This shows us that $h'(y) = 0$, and therefore we can take $h(y) = 0$. It follows that

$$\psi(x, y) = x^3 + x^2 y.$$

The family of solutions that we obtain is therefore given by

$$x^3 + x^2 y = C.$$

4. (10 points)

(a) (4 points) Find a particular solution to the equation

$$y'' + 2y' + y = \cos(5t)$$

(b) (2 points) Find a particular solution to the equation

$$y'' + 2y' + y = \sin(5t)$$

(c) (2 points) Find a particular solution to the equation

$$y'' + 2y' + y = 3\cos(5t) - \sin(5t)$$

(d) (2 points) Write down the general solution to the equation

$$y'' + 2y' + y = 3\cos(5t) - \sin(5t)$$

Solution 4.

(a) Squigglifying, we obtain

$$\widetilde{y}'' + 2\widetilde{y}' + \widetilde{y} = e^{5it}.$$

To solve this, we try a particular solution

$$\widetilde{y}_p = Ae^{5it}.$$

Then

$$\widetilde{y}_p' = 5iAe^{5it}, \quad \widetilde{y}_p'' = -25Ae^{5it}.$$

Putting this all into the original equation, we find

$$(-24 + 10i)Ae^{5it} = e^{5it}.$$

Therefore

$$A = \frac{1}{-24 + 10i} = \frac{-24}{676} - \frac{10}{676}i.$$

It follows that

$$\begin{aligned}
\widetilde{y}_p &= \left(\frac{-24}{676} - \frac{10}{676}i\right) e^{5it} \\
&= \left(\frac{-24}{676} - \frac{10}{676}i\right) (\cos(5t) + i\sin(5t)) \\
&= \left(\frac{-24}{676}\cos(5t) + \frac{10}{676}\sin(5t)\right) + i\left(\frac{-10}{676}\cos(5t) + \frac{-24}{676}\sin(5t)\right)
\end{aligned}$$

Therefore

$$y_p = \mathrm{Re}(\widetilde{y}_p) = \frac{-24}{676}\cos(5t) + \frac{10}{676}\sin(5t)$$

(b)

$$y_p = \mathrm{Im}(\widetilde{y}_p) = \frac{-10}{676}\cos(5t) + \frac{-24}{676}\sin(5t)$$

(c)

$$y_p = 3\mathrm{Re}(\widetilde{y}_p) - \mathrm{Im}(\widetilde{y}_p) = \frac{-62}{676}\cos(5t) + \frac{54}{676}\sin(5t)$$

(d)

$$y = (At + B)e^{-t} + \frac{-62}{676}\cos(5t) + \frac{54}{676}\sin(5t)$$

5. (10 points) Given that $y_1 = t^2$ is a solution to the differential equation

$$t^2 y'' - 3ty' + 4y = 0,$$

use the method of reduction of order to find the general solution of the equation.

Solution 5. We substitute $y = y_1 z = t^2 z$. Then $y' = 2tz + t^2 z'$ and $y'' = 2z + 4tz' + t^2 z''$. Putting this all into the original equation, we find

$$t^4 z'' + t^3 z' = 0.$$

Now substituting $w = z'$, we obtain the first-order separable equation

$$t^4 w' + t^3 w = 0.$$

The solution to this is $w = Ct^{-1}$. Therefore

$$z = \int w dt = C \ln |t| + B.$$

Thus

$$y = t^2 z = Ct^2 \ln |t| + Bt^2$$

is the general solution to the equation.

6. (20 points) A mass of 3 kg is connected to a spring, stretching it 2.4525 meters to a new equilibrium length. The spring is then stretched downward $\sqrt{2}/2$ meters and then thrown downward at a velocity of 1 meter per second. (Note: gravitational acceleration $g = 9.81$ m/s^2)

   (a) (5 points) Set up an initial value problem describing the position of the spring relative to its equilibrium position $u(t)$.

   (b) (5 points) Solve the IVP of (a) to find $u(t)$.

   (c) (5 points) Determine the amplitude, period, and phase of the resultant motion.

   (d) (5 points) Sketch a graph of $u(t)$, and indicate how the amplitude, period, and phase affect the shape of the graph.

Solution 6.

(a) $m = 3$, $k = m \cdot g/l = 3 \cdot 9.81/2.4525 = 12$. Therefore

$$3u'' + 12u = 0, \quad u(0) = \sqrt{2}2, u'(0) = 1.$$

(b) The general solution to the previous equation is

$$u(t) = A\sin(2t) + B\cos(2t).$$

The initial conditions give us $B = \sqrt{2}/2, A = 1/2$. Therefore

$$u(t) = \frac{1}{2}\sin(2t) + \frac{\sqrt{2}}{2}\cos(2t).$$

(c) Amplitude is $R = \sqrt{3}2$, period is $T = 2\pi/\omega = \pi$, phase is $\delta = \arctan\sqrt{2}/2 \approx 0.61548$.

$$u(t) = \sqrt{3}2\cos(2t - 0.61548).$$

(d) You should be able to do this! Note that the phase pushes it to the right by $\delta/\omega = 0.30774$.