Weekly Homework 2

Due: Wednesday April 22, 2015

April 15, 2015

Problem 1 Existence and Uniqueness of Solutions to Linear Equations. For each of the following first order linear initial value problems, determine the largest open inverval on which we should expect there to be a unique solution.

(a) $y' = \sin(x)y + \cot(x)$, $y(\pi/2) = 3$

(b) $xy' + 3y = x^2$, $y(1) = 0$

(c) $y' = y/x + x\tan(x)$, $y(\pi/4) = 1$

Problem 2 Existence and Uniqueness of Solutions to Nonlinear Equations. For each of the following initial value problems, determine with justification which of the following hold

(i) no solution exists

(ii) a unique solution exists

(iii) multiple solutions exist

(a) $y' = y^{1/3}, y(1) = 0$

(b) $y' = y^{1/3}, y(0) = 1$

(c) $yy' = 1/x, y(0) = 1$

Problem 3 Fluid Mixing. A 1000 gallon holding tank that catches runoff from some chemical process initially has 800 gallons of water with 2 ounces of pollution dissolved in it. Polluted water flows into the tank at a rate of 3 gal/hr and contains 5 ounces/gal of pollution in it. A well mixed solution leaves the tank at 3 gal/hr as well. When the amount of pollution in the holding tank reaches 500 ounces the inflow of polluted water is cut off and fresh water will enter the tank at a decreased rate of 2 gallons while the outflow is increased to 4 gal/hr. Determine the amount of pollution in the tank at any time t.

Problem 4 More Fluid Mixing. Initially, a mass of ten grams of salt is dissolved in a 10 liter tank full of water. Then water containing salt at a concentration of 10 grams per liter trickles in at a rate of two liters per hour. A well-mixed solution trickles out at a rate of 3 liters per hour. Find the concentration (in grams per liter) of the salt in the tank at the time when the tank contains 4 liters.

Problem 5 Monetary Investment. A young person with no initial capital invests k dollars per year at an annual rate of return r. Assume that investments are made continuously and that the return is compounded continuously.

(a) Determine the sum $S(t)$ accumulated at any time t

(b) If $r = 7.5\%$ determine k so that 1 million will be available for retirement in 40 years

(c) If $k = 2000$ per year, determine the return rate r that must be obtained to have 1 million available in 40 years

Problem 6 More Fluid Mixing. A 1500 gallon tank initially contains 600 gallons of water with 5 lbs of salt dissolved in it. Water enters the tank at a rate of 9 gal/hr and the water entering the tank has a salt concentration of $\frac{1}{5}(1 + \cos(t))$ lbs/gal. If a well mixed solution leaves the tank at a rate of 6 gal/hr, how much salt is in the tank when it overflows?

Problem 7 Challenger Deep. An intrepid research team plans to explore the Challenger Deep, located at the southern end of the Mariana Trench. The ocean floor is as deep as 10.916 kilometers, making it the deepest point in the ocean floor. (In comparison, the avrage depth of the ocean is 3.688 kilometers)

The research team will pilot a spherical vessel with a radius r and mass m. The force of gravity will do the work in bringing the vessel to the bottom. For this problem, you may make the following assumptions

(i) the density of ocean water is $\rho = 1027$ kg/m^3

(ii) the gravitational acceleration is $g = 9.81$ m/s^2

(iii) the dynamic viscosity of ocean water is $\mu = 1.88 \times 10^{-3}$ kg/(m$\cdot s$)

(iv) the force of drag satisfies Stokes law $F_D = -6\pi\mu r v$, where v is the flow velocity

(v) the crew are not attacked by a sea monster on the descent

With this in mind, answer the following questions

(a) find an equation, in terms of r and m, for how long it takes the vessel to reach the ocean floor

(b) if $r = 1.1$ meters, and m is 11.8 tonnes, how long will the descent take?

Problem 8 Jean Wilder's Famous Problem. A population of Oompa Loompas in a region will grow at a rate that is proportional to their current population. In the absence of any outside factors the population will triple in two weeks time. Also on any given day there is a net migration into the area of 15 Oompa Loompas and 16 are eaten by Wangdoodles, Hornswogglers, Snozzwangers and rotten, Vermicious Knids and 7 die of natural causes. If there are initially 100 Oompa Loompas in the area, will the population survive? If not, when do they die out?

Problem 9 Bernoulli Equations. A Bernoulli equation is a nonlinear equation of the form

$$y' + p(t)y = q(t)y^n$$

If $n \neq 0$ and $n \neq 1$, then substituting $u = y^{1-n}$ and differentiating yeilds

$$u' = (1-n)y^{-n}y'.$$

This tells us that $y' = \frac{y^n}{(1-n)}u'$. Putting this back into the original differential equation then says

$$\frac{y^n}{1-n}u' + p(t)y = q(t)y^n.$$

Dividing both sides by y, we then get

$$\frac{y^{n-1}}{1-n}u' + p(t) = q(t)y^{n-1}.$$

Now if we notice that $y^{n-1} = 1/u$, then this means

$$\frac{1/u}{1-n}u' + p(t) = q(t)(1/u),$$

which simplifies to

$$\frac{1}{1-n}u' + p(t)u = q(t),$$

which is a linear equation in u. We've just made a nonlinear equation into a linear equation... a small miracle. We can then solve for u, and then use the fact that $u = y^{1-n}$ to obtain y. Let's call this method "Bernoulli's method".

(a) Use Bernoulli's method to solve the differential equation

$$y' = (\Gamma \cos(t) + T)y - y^3$$

where here Γ and T are constants. This equation comes up in the study of stability in fluid flows.

Problem 10 Norton's Dome. Norton's Dome is a radially symmetric surface whose height above the ground is of the form

$$h(r) = -\frac{2K}{3g}r^{3/2}$$

where r is the radial distance from the center of the dome and our coordinate system is chosen so that the top of the dome has height $h = 0$. here K is a proportionality factor, so that (K/g) has units of length. Set a point mass on top of the dome and let it slide down from the force of gravity, assuming that there are no friction forces. From the laws of classical mechanics, the radial position $r(t)$ of the point mass may be shown to satisfy the initial value problem

$$r'' = K\sqrt{r}, \quad r(0) = 0, \quad r'(0) = 0.$$

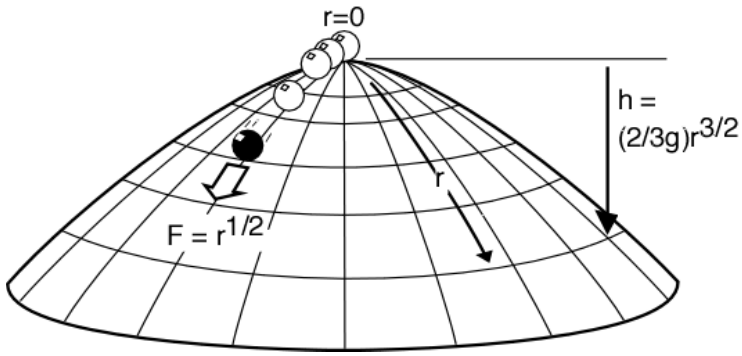

Figure 1: A picture of Norton's Dome.

(a) Show that $r(t) = K^2t^4/144$ is a solution to the initial value problem

(b) Show that $r(t) = 0$ is also a solution to the initial value problem

In fact, for any $\ell \geq 0$

$$r(t) = \begin{cases} 0, \ t < \ell \\ K^2(t-\ell)^4/144, \ t \geq \ell \end{cases}$$

is also a solution to the initial value problem (you need not show this). This is an example of what is called non-determinism in classical mechanics. There is no unique solution to the differential equation: from the point of view of mathematics, the point particle could simply sit there for all eternity, or it could sit there for some arbitrary amount of time and then suddenly roll off for no particular reason! To learn more about this, try googling Norton's Dome.