MATH 307: Problem Set #2

Due on: April 5, 2013

Problem 1 Existence and Uniqueness of Solutions to Linear Equations

For each of the following first order linear initial value problems, determine the largest open inverval on which we should expect there to be a unique solution.

(a) $y' = \sin(x)y + \cot(x)$, $y(\pi/2) = 3$

(b) $xy' + 3y = x^2$, $y(1) = 0$

(c) $y' = y/x + x\tan(x)$, $y(\pi/4) = 1$

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Solution 1.

(a) $(0, \pi)$

(b) $(0, \infty)$

(c) $(0, \pi/2)$

Problem 2 Existence and Uniqueness of Solutions to Nonlinear Equations

For each of the following initial value problems, determine with justification which of the following hold

(i) no solution exists

(ii) a unique solution exists

(iii) multiple solutions exist

(a) $y' = y^{1/3}, y(1) = 0$

(b) $y' = y^{1/3}, y(0) = 1$

(c) $yy' = 1/x, y(0) = 1$

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Solution 2.

(a) By separating the equation, we get the solutions $y = \pm\sqrt{\left(\frac{2}{3}x - \frac{2}{3}\right)^3}$, therefore multiple solutions exist.

(b) Note that for $f(x, y) = y^{1/3}$, $f_y(x, y) = \frac{1}{3}y^{-2/3}$. Both $f(x, y)$ and $f_y(x, y)$ are well-behaved in the open rectangle

$$R = \{(x, y) : -42 < x < 42, \quad 0 < y < 2\}$$

and therefore by the existence/uniqueness theorem for nonlinear first order equations, there exists a unique solution to the IVP in some open interval (a, b) with $a < 0 < b$.

(c) This differential equation has no solution! To see this, suppose that it has a solution – call it y. Then $z = y^2$ satisfies

$$z' = 2yy' = 2/x, \ z(0) = 1.$$

However by the fundamental theorem of calculus, the only solutions to $z' = 2/x$ are of the form $z = \ln|x| + C$, and for no value of C does $z(0) = 1$. Therefore y cannot exist.

Problem 3 Fluid Mixing

A 1000 gallon holding tank that catches runoff from some chemical process initially has 800 gallons of water with 2 ounces of pollution dissolved in it. Polluted water flows into the tank at a rate of 3 gal/hr and contains 5 ounces/gal of pollution in it. A well mixed solution leaves the tank at 3 gal/hr as well. When the amount of pollution in the holding tank reaches 500 ounces the inflow of polluted water is cut off and fresh water will enter the tank at a decreased rate of 2 gallons per hour while the outflow is increased to 4 gal/hr. Determine the amount of pollution in the tank at any time t.

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Solution 3. We will use P to represent the amount of pollutant (in ounces) in the tank, t the time (in hours), and V the volume of liquid in the tank (in gallons). Initially, the differential equation for the amount P of pollutant in the tank is given by

$$\frac{dP}{dt} = \overbrace{3\frac{\text{gal}}{\text{hr}} \cdot 5\frac{\text{oz}}{\text{gal}}}^{\text{rate in}} - \overbrace{3\frac{\text{gal}}{\text{hr}} \cdot \frac{P}{V}\frac{\text{oz}}{\text{gal}}}^{\text{rate out}}.$$

and satisfies the initial condition that $P(0) = 2$ ounces. Note that the rate in of liquid is equal to the rate out of liquid during the first time period, and therefore $V = V(0) = 800$. Thus the initial value proble we must solve is

$$\frac{dP}{dt} = 15 - \frac{3}{800}P, \; P(0) = 2.$$

The differential equation is separable, and solving it we find

$$P = 4000 + Ce^{-(3/800)t}.$$

The initial condition then tells us that $C = -3998$, and consequently

$$P(t) = 4000 - 3998e^{-(3/800)t}.$$

Next we wish to find the time when the amount of pollutant in the tank is 500 ounces of pollutant. To do so, we set $P(t) = 500$ and solve for t:

$$500 = 4000 - 3998e^{-(3/800)t} \;\Rightarrow\; -(3/800)t = -\frac{800}{3}\ln\frac{3500}{3998} \approx 35.475 \text{ hours}.$$

After this time, the situation in the tank changes. The inflow of pollutant is shut off, and instead fresh water is let in. The inflow rate is no longer the outflow rate, so the volume is not constant. In fact, the initial value problem describing the volume is

$$\frac{dV}{dt} = 2\frac{\text{gal}}{\text{hr}} - 4\frac{\text{gal}}{\text{hr}}, \; V(35.475) = 800.$$

Therefore, we find $V(t) = 800 - 2(t - 35.475)$. The initial value problem for the amount of pollutant in the tank as a function of time is then

$$\frac{dP}{dt} = \overbrace{2\frac{\text{gal}}{\text{hr}} \cdot 0\frac{\text{oz}}{\text{gal}}}^{\text{rate in}} - \overbrace{4\frac{\text{gal}}{\text{hr}} \cdot \frac{P}{V}\frac{\text{oz}}{\text{gal}}}^{\text{rate out}},$$

with the initial condition that $P(35.475) = 500$ ounces. Therefore we must solve the initial value problem

$$\frac{dP}{dt} = -4\frac{P}{800 - 2(t - 35.475)}, \; P(35.475) = 500.$$

Again this is a separable equation, and solving it, we obtain

$$P(t) = C(800 - 2(t - 35.475))^2$$

and the initial condition tells us that $C = 500/800$. Thus as our final answer for the amount of pollutant in the tank as a function of time is

$$P(t) = \begin{cases} 4000 - 3998e^{-(3/800)t}, & t \le 35.475 \\ \frac{500}{800}(800 - 2(t - 35.475))^2, & t > 35.475 \end{cases}$$

Problem 4 More Fluid Mixing

Initially, a mass of ten grams of salt is dissolved in a 10 liter tank full of water. Then water containing salt at a concentration of 10 grams per liter trickles in at a rate of two liters per hour. A well-mixed solution trickles out at a rate of 3 liters per hour. Find the concentration (in grams per liter) of the salt in the tank at the time when the tank contains 4 liters.

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Solution 4. We will let S be the amount of salt in the tank as a function of time (in grams), and Q be the concentration of salt in the tank (in grams/liter), and V be the volume of water in the tank (in liters), and t time (in hours). Since the rate in of liquid is different from the rate out, we know that V is not constant. In fact V satisfies the IVP

$$\frac{dV}{dt} = 2 - 3,\ V(0) = 10.$$

Solving this, we find $V(t) = 10 - t$. Next, we set up a differential equation for the amount of salt S in the tank as a function of time. We see that

$$\frac{dS}{dt} = \overbrace{2\frac{\text{ltr}}{\text{hr}} \cdot 10\frac{\text{g}}{\text{ltr}}}^{\text{rate in}} - \overbrace{3\frac{\text{ltr}}{\text{hr}} \cdot \frac{S}{V}\frac{\text{g}}{\text{ltr}}}^{\text{rate out}},$$

with the initial condition othat $S(0) = 10$. Therefore the initial value problem we must solve is

$$\frac{dS}{dt} = 20 - \frac{3}{10-t}S, \quad S(0) = 10.$$

This equation is not separable, but it is linear, so we can solve it with an integrating factor. We calculate

$$\mu(t) = e^{\int \frac{3}{10-t}dt} = e^{-3\ln(10-t)} = (10-t)^{-3}.$$

Multiplying the original differential equation by μ, we get the exact equation

$$(10-t)^{-3}\frac{dS}{dt} = 20(10-t)^{-3} - 3(10-t)^{-4}S.$$

We then move the S-terms over to the left hand side, group, and integrate:

$$\begin{aligned} (10-t)^{-3}S' + 3(10-t)^{-4}S &= 20(10-t)^{-3} \\ ((10-t)^{-3}S)' &= 20(10-t)^{-3} \\ (10-t)^{-3}S &= 10(10-t)^{-2} + C \\ S &= 10(10-t) + C(10-t)^3 \end{aligned}$$

Now the initial condition tells us $C = -9/100$, and therefore

$$S = 10(10-t) - \frac{9}{100}(10-t)^3.$$

The concentration as a function of time is therefore $Q = S/V$, giving us

$$Q = \frac{S}{V} = 10 - \frac{9}{100}(10-t)^2.$$

The tank reaches a volume of 4 liters after exactly 6 hours. The concentration at this time is then seen to be

$$Q(6) = 10 - \frac{9}{100}(10-6)^2 = \frac{214}{25} \approx 8.56 \text{ g per liter}.$$

Problem 5 Monetary Investment

A young person with no initial capital invests k dollars per year at an annual rate of return r. Assume that investments are made continuously and that the return is compounded continuously.

(a) Determine the sum $S(t)$ accumulated at any time t

(b) If $r = 7.5\%$ determine k so that 1 million will be available for retirement in 40 years

(c) If $k = 2000$ per year, determine the return rate r that must be obtained to have 1 million available in 40 years

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Solution 5.

(a) Let t be time in years. Then S satisfies the differential equation

$$\frac{dS}{dt} = rS + k,$$

which has the integrating factor $\mu(t) = e^{-rt}$. Using this to solve:

$$\begin{aligned}
-re^{-rt}S + e^{rt}\frac{dS}{dt} &= ke^{-rt} \\
\left(e^{-rt}S\right)' &= ke^{-rt} \\
\int \left(e^{-rt}S\right)' dt &= \int ke^{-rt} dt \\
e^{-rt}S &= -\frac{k}{r}e^{-rt} + C \\
S &= -\frac{k}{r} + Ce^{rt}
\end{aligned}$$

Since there is no initial capital, $S(0) = 0$, and therefore $C = k/r$, making

$$S = \frac{k}{r}\left(e^{rt} - 1\right)$$

(b) Given that $r = 0.075$, we want $S(40) = 10^6$. Solving for k we obtain

$$\begin{aligned} 10^6 &= \frac{k}{0.075}\left(e^{0.075*40} - 1\right) \\ 10^6 &= 254.474k \\ k &= 3929.68 \text{ dollars per year} \end{aligned}$$

(c) Given that $k = 2000$, we want $S(40) = 10^6$. Solving for r we obtain

$$\begin{aligned} 10^6 &= \frac{2000}{r}\left(e^{r*40} - 1\right) \\ r &= 0.097734 \end{aligned}$$

So we'd need a rate of 9.7734%.

Problem 6 More Fluid Mixing

A 1500 gallon tank initially contains 600 gallons of water with 5 lbs of salt dissolved in it. Water enters the tank at a rate of 9 gal/hr and the water entering the tank has a salt concentration of $\frac{1}{5}(1 + \cos(t))$ lbs/gal. If a well mixed solution leaves the tank at a rate of 6 gal/hr, how much salt is in the tank when it overflows?

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Solution 6. Notice first of all that more water enters the tank than leaves the tank. In fact, the volume satisfies the differential equation

$$\frac{dV}{dt} = \overbrace{9}^{\text{gal/hr in}} - \overbrace{6}^{\text{gal/hr out}},$$

and therefore $\frac{dV}{dt} = 3$, so that $V = 3t + V_0$, where V_0 is the initial volume ($V_0 = 600$). Thus

$$V = 3t + 600.$$

The weight W of of salt in the tank (in pounds) satisfies the differential equation

$$\frac{dW}{dt} = \text{rate in} - \text{rate out},$$

where

$$\text{rate in} = \overbrace{\frac{1}{5}(1 + \cos(t))}^{\text{lbs salt/gallon in}} \times \overbrace{9}^{\text{gal/hr in}}$$

and

$$\text{rate out} = \overbrace{\frac{W}{V}}^{\text{lbs salt/gallon out}} \times \overbrace{6}^{\text{gal/hr out}}.$$

Thus

$$\frac{dW}{dt} = \frac{9}{5}(1 + \cos(t)) - \frac{2W}{t + 200}$$

This equation is linear! An integrating factor is $\mu(t) = (t+200)^2$. Using this to solve, we get

$$\begin{aligned}
2(t + 200)W + (t + 200)^2\frac{dW}{dt} &= \frac{9}{5}(1 + \cos(t))(t + 200)^2 \\
((t + 200)^2 W)' &= \frac{9}{5}(1 + \cos(t))(t + 200)^2 \\
\int ((t + 200)^2 W)' dt &= \int \frac{9}{5}(1 + \cos(t))(t + 200)^2 dt \\
(t + 200)^2 W &= \frac{9}{5}(t + 200)^2 \sin(t) + \frac{18}{5}(t + 200)\cos(t) \\
&\quad - \frac{18}{5}\sin(t) + \frac{9}{5}(t + 200)^3 + C
\end{aligned}$$

so that

$$W = \frac{9}{5}\sin(t) + \frac{18}{5}\frac{\cos(t)}{t + 200} - \frac{18}{5}\frac{\sin(t)}{(t + 200)^2} + \frac{9}{5}(t + 200) + C$$

Sinc initially there are 5 pounds of salt dissolved in the tank $W(0) = 5$, so that

$$5 = \frac{18}{5}\frac{1}{200} + \frac{9}{5}(200) + C$$

and therefore $C = -355.018$, making

$$W = \frac{9}{5}\sin(t) + \frac{18}{5}\frac{\cos(t)}{t + 200} - \frac{18}{5}\frac{\sin(t)}{(t + 200)^2} + \frac{9}{5}(t + 200) - 355.018$$

Now from our equation for V, we know that the tank overflows at $t = 300$. Evaluating $W(300)$, we obtain

$$W(300) = 543.182 \text{ lbs}$$

Problem 7 Challenger Deep

An intrepid research team plans to explore the Challenger Deep, located at the southern end of the Mariana Trench. The ocean floor is as deep as 10.916 kilometers, making it the deepest point in the ocean floor. (In comparison, the avrage depth of the ocean is 3.688 kilometers)

The research team will pilot a spherical vessel with a radius r and mass m. The force of gravity will do the work in bringing the vessel to the bottom. For this problem, you may make the following assumptions

(i) the density of ocean water is $\rho = 1027$ kg/m^3

(ii) the gravitational acceleration is $g = 9.81$ m/s^2

(iii) the dynamic viscosity of ocean water is $\mu = 1.88 \times 10^{-3}$ kg/(m$\cdot s$)

(iv) the force of drag satisfies Stokes law $F_D = -6\pi\mu r v$, where v is the flow velocity

(v) the crew are not attacked by a sea monster on the descent

With this in mind, answer the following questions

(a) find an equation, in terms of r and m, for how long it takes the vessel to reach the ocean floor

(b) if $r = 1.1$ meters, and m is 11.8 tonnes, how long will the descent take?

Solution 7.

(a) Newton tells us $F = ma$, where m is the mass of the sub and $a = v'$ is the acceleration. The forces the sub experiences are gravitational force

$$F_g = -mg,$$

bouyant force

$$F_b = \frac{4}{3}\pi r^3 \rho g,$$

and the force of drag, which according to Stokes law is

$$F_d = -6\pi\mu r v.$$

Therefore

$$mv' = \frac{4}{3}\pi r^3 \rho g - mg - 6\pi\mu r v.$$

This is a separable equation in v: the solution is

$$v = Ce^{-\frac{6\pi\mu r}{m}t} + \frac{-mg + \frac{4}{3}\pi r^3 \rho g}{6\pi\mu r}.$$

The initial velocity is 0, so

$$v = \frac{-mg + \frac{4}{3}\pi r^3 \rho g}{6\pi\mu r}\left(1 - e^{-\frac{6\pi\mu r}{m}t}\right).$$

The y-position as a function of time is the integral of v, and the initial y position is 0, so

$$y = \frac{-mg + \frac{4}{3}\pi r^3 \rho g}{6\pi\mu r\left(1 + \frac{m}{6\pi\mu r}\right)}\left(t - e^{-\frac{6\pi\mu r}{m}t}\right).$$

The vessel is at the bottom when $y = 10,916$ meters, and therefore the time t_f when it hits the bottom is obtained from solving

$$-10916 = \frac{-mg + \frac{4}{3}\pi r^3 \rho g}{6\pi\mu r\left(1 + \frac{m}{6\pi\mu r}\right)}\left(t_F - e^{-\frac{6\pi\mu r}{m}t_F}\right).$$

This cannot be solved for t_F by hand – we have to use a computer.

(b) Based on the previous equation, the descent time is approximately 2163 seconds, which is approximately 36 minutes. This descent time is faster than it would be in reality because the linear model that we used for the drag is inaccurate when the Reynolds number $\mathrm{Re} = \rho v L/\mu$ is large, as it is in this situation. We should have instead used a quadratic model for the drag force to get more physically accurate results.

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Problem 8 Jean Wilder's Famous Problem

A population of Oompa Loompas in a region will grow at a rate that is proportional to their current population. In the absence of any outside factors the population will triple in two weeks time. Also on any given day there is a net migration into the area of 15 Oompa Loompas and 16 are eaten by Wangdoodles, Hornswogglers, Snozzwangers and rotten, Vermicious Knids and 7 die of natural causes. If there are initially 100 Oompa Loompas in the area, will the population survive? If not, when do they die out?

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Solution 8. Let n_O be the number of Oompa Loopas in our region, and let t be time in days. Then they satisfy the differential equation

$$\frac{dn_O}{dt} = \text{increase} - \text{decrease}$$

where

$$\text{increase} = \overbrace{rn_O}^{\text{growth due to breeding}} + \overbrace{15}^{\text{growth from migration}}$$

and

$$\text{decrease} = \overbrace{16}^{\text{killed by terrifying monsters}} + \overbrace{7}^{\text{die of natural causes}} .$$

Here r is the growth rate. Thus

$$\frac{dn_O}{dt} = rn_O - 8.$$

An integrating factor for this solution is $\mu(t) = e^{-rt}$. Using this to solve, we get

$$\begin{aligned} -re^{-rt}n_O + e^{-rt}\frac{dn_O}{dt} &= -8e^{-rt} \\ (e^{-rt}n_O)' &= -8e^{-rt} \\ \int (e^{-rt}n_O)'dt &= \int -8e^{-rt}dt \\ e^{-rt}n_O &= \frac{8}{r}e^{-rt} + C \\ n_O &= \frac{8}{r} + Ce^{rt} \end{aligned}$$

Since $n_O(0) = 100$, we know that $C = 100 - \frac{8}{r}$, and therefore

$$n_O = \frac{8}{r} + \left(100 - \frac{8}{r}\right)e^{rt}.$$

What is r, though? Outside external influences (such as birth, death, predator interaction, and migration), the population would satisfy the IVP

$$n_O' = rn_0, \quad n_0(0) = 100,$$

which has the solution $n_O = 100e^{rt}$. From the question, we know that in this case the population should triple in 14 days (2 weeks), so that $n_O(14) = 300$. Thus $300 = 100e^{14r}$, making $r = \ln(3)/14$. Thus the population actually satisfies

$$n_O = \frac{8}{\ln(3)/14} + \left(100 - \frac{8}{\ln(3)/14}\right)e^{rt}.$$

or approximately

$$n_O = 101.947 - 1.947e^{\frac{\ln(3)}{14}t}.$$

This is decreasing, so the population dies out! Setting $n_O = 0$ and solving for t, we find that no more Oompa Loompas are left after about 50.442 days.

Problem 9 Bernoulli Equations

A Bernoulli equation is a nonlinear equation of the form

$$y' + p(t)y = q(t)y^n$$

If $n \neq 0$ and $n \neq 1$, then substituting $u = y^{1-n}$ and differentiating yeilds

$$u' = (1-n)y^{-n}y'.$$

This tells us that $y' = \frac{y^n}{(1-n)}u'$. Putting this back into the original differential equation then says

$$\frac{y^n}{1-n}u' + p(t)y = q(t)y^n.$$

Dividing both sides by y, we then get

$$\frac{y^{n-1}}{1-n}u' + p(t) = q(t)y^{n-1}.$$

Now if we notice that $y^{n-1} = 1/u$, then this means

$$\frac{1/u}{1-n}u' + p(t) = q(t)(1/u),$$

which simplifies to

$$\frac{1}{1-n}u' + p(t)u = q(t),$$

which is a linear equation in u. We've just made a nonlinear equation into a linear equation... a small miracle. We can then solve for u, and then use the fact that $u = y^{1/n}$ to obtain y. Let's call this method "Bernoulli's method".

(a) Use Bernoulli's method to solve the differential equation

$$y' = (\Gamma\cos(t) + T)y - y^3$$

where here Γ and T are constants. This equation comes up in the study of stability in fluid flows.

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Solution 9.

(a) We do the substitution $u = y^{-2}$, so that

$$u' = -2y^{-3}y'$$

and

$$y' = -\frac{1}{2}y^3u'$$

Substituting this into the original differential equation for y', we get

$$-\frac{1}{2}y^3u' = (\Gamma\cos(t) + T)y - y^3.$$

Dividing through by y^3 on both sides, this becomes

$$-\frac{1}{2}u' = (\Gamma\cos(t) + T)y^{-2} - 1.$$

Now remembering $u = y^{-2}$:

$$-\frac{1}{2}u' = (\Gamma\cos(t) + T)u - 1.$$

which simplifies to

$$\frac{1}{2}u' + (\Gamma\cos(t) + T)u = 1.$$

This equation is linear! An integrating factor for this equation is

$$\mu(t) = \exp\left(2\Gamma\sin(t) + 2Tt\right),$$

and using this to solve the equation, we get

$$u = \frac{1}{\exp\left(2\Gamma\sin(t) + 2Tt\right)} \int \exp\left(2\Gamma\sin(t) + 2Tt\right) dt$$

so that

$$y = \sqrt{\frac{\exp\left(2\Gamma\sin(t) + 2Tt\right)}{\int \exp\left(2\Gamma\sin(t) + 2Tt\right) dt}}$$

Problem 10 Norton's Dome

Norton's Dome is a radially symmetric surface whose height above the ground is of the form

$$h(r) = -\frac{2K}{3g}r^{3/2}$$

where r is the radial distance from the center of the dome and our coordinate system is chosen so that the top of the dome has height $h = 0$. here K is a proportionality factor, so that (K/g) has units of length. Set a point mass on top of the dome and

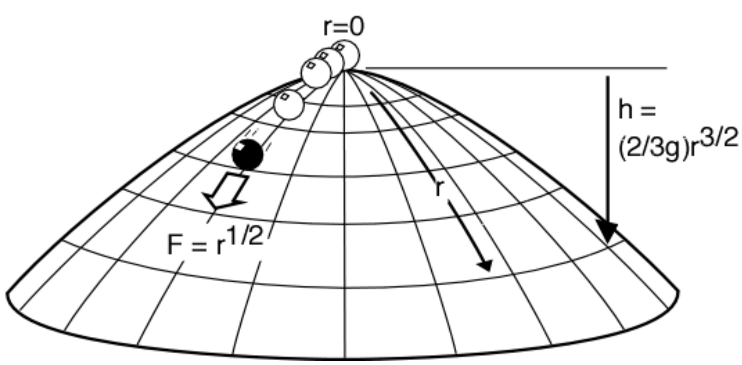

Figure 1: A picture of Norton's Dome.

let it slide down from the force of gravity, assuming that there are no friction forces. From the laws of classical mechanics, the radial position $r(t)$ of the point mass may be shown to satisfy the initial value problem

$$r'' = K\sqrt{r}, \quad r(0) = 0, \quad r'(0) = 0.$$

(a) Show that $r(t) = K^2t^4/144$ is a solution to the initial value problem

(b) Show that $r(t) = 0$ is also a solution to the initial value problem

In fact, for any $\ell \geq 0$

$$r(t) = \begin{cases} 0, \ t < \ell \\ K^2(t-\ell)^4/144, \ t \geq \ell \end{cases}$$

is also a solution to the initial value problem (you need not show this). This is an example of what is called non-determinism in classical mechanics. There is no unique solution to the differential equation: from the point of view of mathematics, the point particle could simply sit there for all eternity, or it could sit there for some arbitrary amount of time and then suddenly roll off for no particular reason! To learn more about this, try googling Norton's Dome.

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Solution 10.

(a) $r''(t) = K^2t^2/12$ and $K\sqrt{r} = K^2t^2/12$, so r satisfies the differential equation. Also, $r(0) = r'(0) = 0$, so it satisfies the IVP.

(b) $r''(t) = 0$ and $K\sqrt{r} = 0$, and $r(0) = r'(0) = 0$, so $r = 0$ also satisfies the IVP.